Lecture 15: Gaussian Elimination. Reduced Row Echelon Form.

Aim Lecture Recall $(A|\mathbf{b})$ easy to solve if A in row echelon form & EROs don't change soln set. Show can always apply

This algorithm called

Gaussian Elimination

E.g. 1 Solve

$$(A|\,\mathbf{b}) = \begin{pmatrix} 0 & 1 & 1 & 2 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 3 & 0 & 2 \\ -1 & 3 & 2 & 3 & 0 & -5 \end{pmatrix}$$

N.B. There are 4 eqns & 5 unknowns so expect

Step 1 Choose a pivot element.

Defn The 1st (=leftmost) non-zero column (i.e.

1st non-zero entry in this

Pivot row is the row

Step 2 Apply ERO switching rows so pivot

$$(A|\,\mathbf{b}) \xrightarrow{R1 \quad R2} \begin{pmatrix} & & & & & \\ & & & & & \\ 1 & 0 & 1 & 3 & 0 & 2 \\ -1 & 3 & 2 & 3 & 0 & -5 \end{pmatrix}$$

Step 3 Add multiples of pivot row from those below until

N.B. this corresponds to elim

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 0 & -1 \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}$$

Step 4 Repeat steps 1-3 for part of matrix below

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 & & 1 \\ 0 & 1 & 1 & 2 & 0 & & -1 \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 & & 1 \\ 0 & 1 & 1 & 2 & 0 & & -1 \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \end{pmatrix} = (U|\mathbf{y})$$

Note U is in

Can now solve using back-substn OR

Reduced Row Echelon Form Let A be a matrix in row echelon form. We say it is in row echelon form if

i) every leading column has only

& ii) all leading

Step 5 Scale rows so leading entries are 1. Add multiples of leading rows to rows above so $(U|\mathbf{y})$ simplifies

$$(U|\mathbf{y}) \xrightarrow{R2 \mapsto R2 -} \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & & 1 \\ 0 & 1 & 1 & & & & \\ 0 & 0 & 0 & 1 & -1 & & 2 \\ 0 & 0 & 0 & 0 & 0 & & 0 \end{pmatrix}$$

cont'd

Non-leading columns corresp

Intro parameters

3rd row $\implies$

2nd row $\implies$

1st row $\implies$

$\mathbf{x} =$

solns form

Warning In gen, avoid performing more than

The exception is during Step 2, you may add multiples of a single pivot row to as many others as you like simultaneously.

Case of No Solns

e.g. 2 $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 2 \end{array}\right)$.

Corresponds to eqn

which has

This type of row in the row echelon form is always the culprit

e.g. 3 Solve

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & -1 & -1 & 1 \\ 4 & 1 & -3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ & & & \\ & & & \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & & & \\ 0 & & & \end{pmatrix}$$

Bottom row

Upshot Let $(A|\,\mathbf{b})$ be a system of lin eqns & $(U|\mathbf{y})$ be a row echelon form obtained by applying

Facts 1 If some row, say i-th row of U is zero (i.e.

& corresponding component y_i of

then

N.B. This is equivalent to saying $\mathbf{y}$ is a

2 If no such row occurs then, back-substn

3 In this case, the no. param in the soln is

e.g. 4 Consider following systems of lin eqns $(U|\mathbf{y})$.

$$\begin{pmatrix} 1 & 2 & 3 & & 4 \\ 0 & 5 & 6 & & 7 \\ 0 & 0 & 9 & & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & & 4 \\ 0 & 5 & 6 & & 7 \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & & 4 \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$