Lecture 14: Row Echelon Form. Elementary Row Operations.

Aim Lecture Key to solving lin eqns involves 2 concepts.

i) Elementary Row Operations (EROs): which

& ii) Row Echelon Form: which are systems of lin eqns, sufficiently

Analyse Easy Example Solve

$$\left.\begin{aligned} x - 2y &= 3 \qquad \ldots (1) \\ x + y &= -3 \qquad \ldots (2) \end{aligned}\right\} (*)$$

Eliminate x: (2) - (

In fact solns to (*) same as solns to

$$\left.\begin{array}{rl} x - 2y = 3 & \dots (1) \\ = & \dots (2) \end{array}\right. \quad (**)$$

Row Echelon Form Consider system of lin eqns $A\mathbf{x} = \mathbf{b}$ with A an $m \times n$-matrix with entries in field $\mathbb{F}$, and $\mathbf{b} \in \mathbb{F}^m$. Need solve for $\mathbf{x} \in$

Q When can you solve for variables, 1 at a time as in (**) above?

A When A is in

First some auxiliary defns.

Leading entry means first (= leftmost) non-

A leading row (or column) is one

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ \pi & -1 & 1 & 1 \end{pmatrix}$$

Defn A matrix A is in row echelon form if

i) all non-leading rows are

ii) as you go down rows, the leading

e.g. Following are not in row echelon form

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

e.g. Following is in row echelon form

Back-Substitution Can solve $(A|\mathbf{b})$ easily using "back-substitution" method below if

e.g. Solve

$$\begin{pmatrix} 1 & 1 & 2 & & 1 \\ 0 & 1 & 1 & & -1 \\ 0 & 0 & 2 & & 4 \end{pmatrix}$$

i.e. $x_1 + x_2 + 2x_3 = 1 \ldots (1)$

$(3) \implies x_3$

$(2) \implies x_2 +$

$(1) \implies x_1$

Hence, soln is $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$

If there are non-leading columns in A, you need to introduce a

e.g. Solve (over the reals)

$$\begin{pmatrix} 1 & -1 & 1 & -1 & & 0 \\ 0 & 0 & 1 & 2 & & 1 \end{pmatrix}$$

N.B. Expect soln given by a plane

Columns 2 & 4 not leading so intro

2nd row $\implies$

1st row $\implies$

Hence, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} =$

Elementary Row Operations

Q How can you alter (simplify) a system $(A|\mathbf{b})$ of lin eqns without

A You can apply any of the following

ERO1 Switch 2 rows of

$$\begin{pmatrix} 1 & 2 & & 3 \\ 4 & 5 & & 6 \\ 7 & 8 & & 9 \end{pmatrix}$$

Same set of solns as this corresp to

ERO2 Multiply a row by

$$\begin{pmatrix} 1 & 2 & & 3 \\ 4 & 4 & & 4 \end{pmatrix}$$

ERO3 Add a multiple of

$$\begin{pmatrix} 1 & 0 & 1 & & 2 \\ -2 & -1 & -1 & & 1 \end{pmatrix}$$

Let's rewrite original example in new notn

$$x - 2y = 3$$

$$x + y = -3$$

$$\begin{pmatrix} 1 & -2 & & 3 \\ & & & -3 \end{pmatrix}$$