Lecture 13: Systems of Linear Eqns

Aim Lecture Solns to many physical problems can be obtained by solving systems of linear equations. We set up a general framework

Some simple examples you know

e.g.1 $x - y = 1$

∞ no. solns

corresponding to

e.g.2

$x - y = 1$

$2x - y = 0$

Soln: $(2) - (1) \implies x =$

There is only

General Defn Let $\mathbb{F}$ be a field, $x_1, \ldots, x_n$ variables. A system of linear equations $(/\mathbb{F})$ is a set

$$
\begin{aligned}
&= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
\qquad (*)
$$

where all a_{ij},

Say system of eqns is homogeneous if

Usually abbreviate (*) using matrix notn as follows.

Defn For integers $m, n \geq 1$, an $m \times n$-matrix (with entries in $\mathbb{F}$) is an array

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} = A$$

where all

N.B. There are m

Usually abbrev above system of lin eqns (*) to augmented

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \\ a_{21} & a_{22} & & a_{2n} & \\ & & & & \\ a_{m1} & a_{m2} & & a_{mn} & \end{pmatrix}$$

so have augmented A

Vectors as matrices An $m \times 1$-matrix

so often denote this as a vector $\mathbf{b}$. Conversely, $\mathbf{b} \in \mathbb{F}^m$ will usually not refer to the row vector $(b_1, \ldots, b_m)$ but rather

Rem Can now rewrite augmented

Lin eqns & "linear" geometry Let's work $/\,\mathbb{R}$ in this geom section. Consider a plane P_1 in $\mathbb{R}^3$ defined by

& sim plane P_2 defined by

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

Solns to

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & |b_1 \\ a_{21} & a_{22} & a_{23} & \end{pmatrix}$$

correspond

3 possible cases occur

case a) Planes are parallel.

E.g. Solve $\begin{pmatrix} 1 & 2 & 3 & |4 \\ 1 & 2 & 3 & \end{pmatrix}$

i.e. $x_1 + 2x_2 + 3x_3 =$

$\Longrightarrow$ contradiction

no

case b) Planes are the same.

E.g. Solve $\begin{pmatrix} 1 & 2 & 3 & |4 \\ 2 & 4 & 6 & \end{pmatrix}$

Here P_1

solns form

case c) (Generic Case) Planes intersect in

Still ∞ no. solns

Let's add a 3rd eqn, in this generic case

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

Assume it defines another plane P_3 i.e.

Have now 3 further cases:

i) $P_3 \supset L$ so solns to 3 eqns

ii) L parallel (but

iii) L not

Upshot When solving lin eqns in x_1, x_2, x_3 answer is either

α) ∞ no. solns corresp to

β)

or γ)

Later we'll generalise this to higher dimensions.

Other reformulations of systems of lin eqns

Consider system of lin eqn / $\mathbb{F}$

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & & \\ & & & & & \\ a_{m1} & a_{m2} & & a_{mn} & & \end{pmatrix} = [A|\mathbf{b}]$$

Vector Eqn System equivalent to vector eqn

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{pmatrix} + \begin{pmatrix} a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{pmatrix} + \ldots + \begin{pmatrix} a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \ldots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

where $\mathbf{a}_1$,

$(A|\mathbf{b})$ is equiv to vector eqn

Matrix eqn For $\mathbf{x} \in \mathbb{F}^n$ and matrix A as above, we define a new vector $A\,\mathbf{x} \in \mathbb{F}^m$ by

$$A\,\mathbf{x} = \begin{pmatrix} \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix}$$

The $(A|\mathbf{b})$ is equivalent to

E.g. Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ on the plane $\mathbf{x} =$

Reformulate in the 4 forms above.

Via Vector eqn: Can you solve

Via System of Lin Eqns: Can you solve

Via Augmented Matrix: Can you solve

Via Matrix eqn: Can you solve