Lecture: Topological Curves

Aim Begin study of top n-folds with 1-folds = curves C.

Classification Problem

Problem Classify all top n-folds ie. give a complete list of top n-folds up to homeo s.t. no two are homeo.

Results a $n=1,2$ classical & we'll cover
b $n=3$ follows from Pirelman's work 2003.
c Known to be logically impossible for $n>3$.

Classification of Curves

Lemma 1 Let X be second countable. Then every open cover $\{X_\alpha\}$ has a countable subcover.

Proof: Let $B_1, B_2, \ldots$ be open sets defining a countable base. Let X_i be any X_α containing B_i, if such exists (else undefined). The defined X_i are countable.

Suff. note X_i cover $\because$ $p \in X \Rightarrow p \in X_\alpha$ for some α & $\{B_i\}$ a base $\Rightarrow$ $p \in B_i$ for some i s.t. $B_i \subseteq X_\alpha$ $\therefore$ X_i defined & $p \in X_i$. $\square$

Thm 1 Any conn. curve C (ie. conn. top 1-fold) is homeomorphic to $\mathbb{R}$ or the circle S^1.

Proof By defn, C has an open cover $\{I_j\}_{j\in J}$ where I_j are homeo. to conn. open subsets of $\mathbb{R}$ i.e. open intervals which in turn are homeo. to $\mathbb{R}$. Lemma $\Rightarrow$ can ass. $J = \mathbb{N}$. If J finite harmless to repeat some I_j so often.

C conn. $\Rightarrow$ can assume I_k intersects $I_{<k} := \bigcup_{j<k} I_j$. Suff. prove

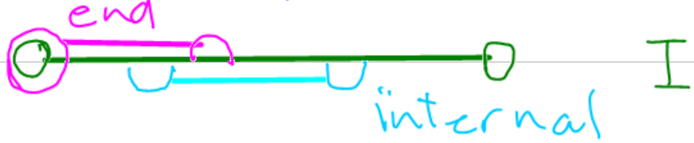
Thm 1' $I_{\le k} := I_1 \cup I_2 \cup \dots \cup I_k$ is homeo. to $\mathbb{R}$ or S^1.

Proof Thm 1' $\Rightarrow$ Thm 1. Suppose $I_{\le k} \simeq S^1$ for some k. Now S^1 compact & C Hausdorff $\Rightarrow$ $I_{\le k} \subseteq C$ closed. Also $I_{\le k}$ open $\xRightarrow{C\ \text{conn}}$ $C = I_{\le k}$.

$\therefore$ Can ass. all $I_{\le k}$ homeo. to open intervals. One easily defines inductively, continuous $f_k : I_{\le k} \to \mathbb{R}$ which are homeo. onto finite open intervals s.t. f_k extends f_{k-1}. Obtain homeo. $f : C \to \mathbb{R}$ onto open subinterval by $f(p) = f_k(p)$ for any k s.t. $p \in I_{\le k}$. $\square$

Sketch proof below in class.

Proof Thm 1' By induction on k, case $k=1$ clear. Given any subinterval I' of an open interval I, we say it's internal if $I - I'$ has 2 conn. comp. & an end if $I - I'$ has one:

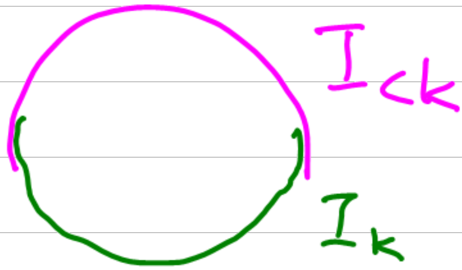


If I' internal then it has 2 distinct bdry pts in I.

We can ass. neither $I_{<k}$ nor I_k is contained in the other, else we're done by induction. Wish to show only two possibilities

I

II



Let I' be a path comp. of $I_{<k} \cap I_k$. (loc. path conn. $\Rightarrow$ I' is an open interval. (Hausdorff $\Rightarrow$ unique limits at either end of I' say p, q (may not exist). Write $I' =: (p,q)$.

suggestive notn to help us remember.

Case 1 $p \notin I_{<k}$ (e.g. doesn't exist). Then $I_{<k} \not\cong S^1$ & I' is an end interval of $I_{<k}$. Can write $I_{<k} = (p,q) \dot\cup [q,b)$ where $[q,b)$ is homeo to a half closed interval. (possibly empty).

$q \in I_{<k} \Rightarrow q \notin I_k$ else $q \in I'$. Hence I' is an end interval of I_k & $I_k = (a,p] \dot\cup (p,q)$ for some half closed interval $(a,p]$

If $(a,p] \cap [q,b) = \emptyset$, then we are in case I, it's easy to show $I_{\leq k}$ homeo to open interval. Else repeat above analysis to see case II holds.

Case 2 $p \in I_{<k}$. Then $p \notin I_k$ & we can swap roles of $I_{<k}$ & I_k (more or less) to arrive to same conclusion. $\square$