Lecture 9: Second Fundamental Form

Aim Lecture Study how par. surface $\sigma : U \twoheadrightarrow S \subseteq \mathbb{R}^3$ bends by studying how Gauss normal $\nu(u)$ varies with u.

Shape Operator L

Differentiate $1 = \langle \nu(u), \nu(u) \rangle$ using product rule in Problem Set 1.

$$0 = \partial_i \langle \nu, \nu \rangle = \langle \partial_i \nu, \nu \rangle + \langle \nu, \partial_i \nu \rangle = 2\langle \nu, \partial_i \nu \rangle$$

mean $\frac{\partial}{\partial u^i}(\langle \nu(u), \nu(u) \rangle$.

Hence

Lemma 1: $\partial_i \nu = d\nu_u(e_i) \in \nu(u)^\perp = T_u S$.

Hence $d\nu_u : T_u U = \mathbb{R}^2 \longrightarrow T_u S$.

Rem By defn, $d\sigma_u : T_u U \longrightarrow T_u \mathbb{R}^3$ is inj. with image $T_u S$ so get invertible linear map $d\sigma_u : T_u U \longrightarrow T_u S$.

Defn 1 (a) For $u \in U$, we define the (1,1)-tensor

$$L(u) = -d\nu_u (d\sigma_u)^{-1} : T_u S \xrightarrow{(d\sigma_u)^{-1}} T_u U \xrightarrow{-d\nu_u} T_u S$$

The shape operator L is the (1,1)-tensor field (guess the defn) $u \mapsto L(u)$.

(b) If G is 1st fund. form, the second fundamental form H is the (0,2)-tensor field $H = G.L$ i.e. $H(u) = G(u).L(u)$.

Eg. 1. If $L = 0$ then $d\nu = 0$ so
ν = const. vector say v.
$\therefore \quad \partial_i \langle v, \sigma \rangle = \langle v, \partial_i \sigma \rangle = 0 \quad , i = 1, 2$
$\therefore$ S lies on plane of form $\langle v, \sigma \rangle$ = const.

Geom. Interpret^n S planar $\Longleftrightarrow$ normal
ν doesn't vary $\Longleftrightarrow L = 0 \Longleftrightarrow H = 0$.

$\boxed{\text{Coordinate tensor for } H}$ work on U
Below we drop variable $u \in U$ from notn.

Formula 1 Coord. tensor

$$H_{ij} = -\langle \partial_i \sigma, \partial_j \nu \rangle \overset{*}{=} \langle \partial_j \partial_i \sigma, \nu \rangle.$$

In particular $\underline{H}$ & $\sigma^* H$ are symm.

Proof: $L\, \partial_j \sigma = -(d\nu \circ d\sigma^{-1})\, d\sigma(e_j)$

$$= -d\nu(e_j) = -\partial_j \nu$$

$$H_{ij} = H(\partial_i \sigma, \partial_j \sigma) = -\langle \partial_i \sigma, L \partial_j \sigma \rangle$$

$$= -\langle \partial_i \sigma, \partial_j \nu \rangle.$$

For * diff. $0 = \langle \partial_i \sigma, \nu \rangle$.

$$0 = \partial_j \langle \partial_i \sigma, \nu \rangle = \langle \partial_j \partial_i \sigma, \nu \rangle + \langle \partial_i \sigma, \partial_j \nu \rangle$$

$$\therefore \quad \langle \partial_i \sigma, \partial_j \nu \rangle = -\langle \partial_j \partial_i \sigma, \nu \rangle.$$

$\square$

Formula 1' $\quad \underline{H} = -(d\sigma)^T(d\nu)$

2×3-matrix $\qquad$ 3×2-matrix.

Proof For $X, Y \in \mathbb{R}^n$

$$X^T \underline{H} Y = X^T \underline{\sigma^* H}\, Y = (\sigma^* H)(X, Y)$$

$$= H(d\sigma(X), d\sigma(Y))$$

$$= \langle d\sigma(X), \underbrace{L}_{-d\nu(d\sigma)^{-1}}(d\sigma)(Y)\rangle$$

$$= -X^T(d\sigma)^T(d\nu)\, Y. \qquad \square$$

$\boxed{\text{Examples}}$

Eg1 Unit speed curve $\gamma(u) = (f(u), g(u))$ with $f(u) > 0$.

Surface of revolution

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))^T$$

Recall from lecture 6

$$\partial_1\sigma = (\dot f(u)\cos v, \dot f(u)\sin v, \dot g(u))^T$$

$$\partial_2\sigma = (-f(u)\sin v, f(u)\cos v, 0)^T$$

$$\nu = (-\dot g(u)\cos v, -\dot g(u)\sin v, \dot f(u))^T$$

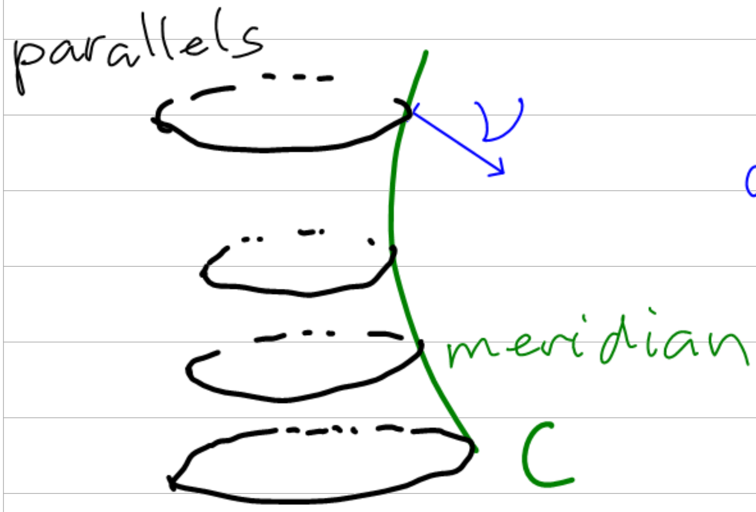
$$\therefore \partial_1\nu = (-\ddot g(u)\cos v, -\ddot g(u)\sin v, \ddot f(u))^T$$

$$\partial_2\nu = (\dot g(u)\sin v, -\dot g(u)\cos v, 0)^T.$$

$$H_{11} = H_{11}(u,v) = -\langle \partial_1\sigma, \partial_1\nu\rangle = \dot f(u)\ddot g(u) - \ddot f(u)\dot g(u)$$

$$= \begin{vmatrix} \dot f & \dot g \\ \ddot f & \ddot g \end{vmatrix}$$

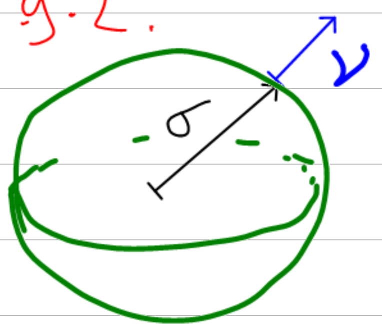
$$H_{12} = H_{21} = -\langle \partial_1\sigma, \partial_2\nu\rangle = 0,$$
$$H_{22} = -\langle \partial_2\sigma, \partial_2\nu\rangle = f(u)\,\dot g(u).$$



$H_{21}=0$ means, as ν moves along meridian C (ie. u changes, v fixed) then ν varies in plane $(\partial_2\sigma)^\perp$ = plane containg C & z-axis.

Sim. moving ν along parallel, it varies in horizontal plane which $\perp$ meridian.

Eg.2.



σ param. sphere S^2 radius R centre $(0,0,0)^T$.

$\therefore\ \nu = \pm \frac{1}{R}\sigma$

Sign depends on orient$^{\underline{n}}$ of param.

$$L = -d\nu\,(d\sigma)^{-1} = \mp \frac{1}{R}\,\mathrm{id}.$$

$$\therefore\ H = -\frac{1}{R}G.$$