Lecture 8: First fundamental form

Aim Lecture Study "inner product" giving metric on surface in terms of param. space U.

(0,2)-tensor fields

Defn 1 A (0,2)-tensor field G on par. n-fold $\sigma : U \xrightarrow{\sim} M \subseteq \mathbb{R}^m$ is a fn assigning

$$\underset{\substack{\cap \\ U}}{u} \longmapsto G(u) \in T_u^{0,2} M := (T_u M)^{0,2} \text{ s.t.}$$

$$* \quad (G(u))\big((\partial_i \sigma)(u), (\partial_j \sigma)(u)\big) \in C^\infty(U) \ \forall i, j.$$

ie, coords wrt $\{\partial_i \sigma\}$ are smooth.

Further, we say G is symmetric if $[G(u)](v, w) = [G(u)](w, v) \quad \forall v, w \in T_u M, u \in U$.

Notn In future abbrev. $[G(u)](v, w) = G(v, w)$.

Consider change of var. $\varphi : \hat{U} \xrightarrow{\sim} U$ & reparam. $\tilde{\sigma} = \sigma \circ \varphi : \tilde{U} \longrightarrow M$.

Jacobian $d\varphi = \left(\frac{\partial u^i}{\partial \tilde{u}^j}\right)$.

$$\partial_i \tilde{\sigma} = \frac{\partial \tilde{\sigma}}{\partial \tilde{u}^i} = \sum_j \frac{\partial \sigma}{\partial u^j} \cdot \frac{\partial u^j}{\partial \tilde{u}^i} = \sum \frac{\partial u^j}{\partial \tilde{u}^i} \partial_j \sigma .$$

writing $u = \varphi(\tilde{u})$ for $\tilde{u} \in \hat{U}$, let $\tilde{G}(\tilde{u}) = G(u)$ & $\tilde{G}_{ij}(\tilde{u})$ be coord. wrt $\{\partial_k \tilde{\sigma}\}$.

$$\tilde{G}_{ij} = \tilde{G}(\partial_i \tilde{\sigma}, \partial_j \tilde{\sigma}) = [\tilde{G}(\tilde{u})]\left(\sum_k \frac{\partial u^k}{\partial \tilde{u}^i} \partial_k \sigma, \sum \frac{\partial u^l}{\partial \tilde{u}^j} \partial_l \sigma\right)$$

$$= \sum_{k,l} \frac{\partial u^k}{\partial \tilde u^i} \frac{\partial u^l}{\partial \tilde u^j} [G(u)] (\partial_k \sigma, \partial_l \sigma)$$

$\Rightarrow$

Formula 1 $\quad \tilde G_{ij}(\tilde u) = \sum_{k,l} \frac{\partial u^k}{\partial \tilde u^i} \frac{\partial u^l}{\partial \tilde u^j} G_{kl}(\underset{\phi(\tilde u)}{\overset{\shortparallel}{u}})$

Note Einstein summation notn.

$\therefore$ **Rem** Smoothness condn $*$ indep. of param.

To declutter notn, we've dropped variable $\tilde u$

Formula 1 $\Longleftrightarrow$

Associated matrix of Lect. 6.

Formula 1' $\quad \underline{\tilde G} = (d\varphi)^T \underline{G}\, d\varphi$

We reprove this using pullback (0,2)-tensor field $\sigma^* G$ on U defined by

$$(\sigma^* G)(\underset{\in}{X}, \underset{\in}{Y}) := G(d\sigma_u(\underset{\in}{X}), d\sigma_u(\underset{\in}{Y}))$$

$T_u U = \mathbb{R}^n \qquad\qquad T_u M$

Fact Let $\underline{\sigma^* G}$ be assoc. matrix wrt $\{e_i\}$.
Then $\underline{\sigma^* G} = \underline{G}$.

Proof $(\sigma^* G)_{ij} = (\sigma^* G)(e_i, e_j) = G(d\sigma(e_i), d\sigma(e_j))$
$= G(\partial_i \sigma, \partial_j \sigma) = G_{ij}$. $\square$

Proof Formula 1' We use Prop Lect. 6 giving, for $X, Y \in \mathbb{R}^n$:

$$X^T \underline{\tilde G} Y \overset{\text{Fact}}{=\!=} X^T \underline{\tilde\sigma^* G} Y \overset{\text{Prop. Lect. 6}}{=\!=} \tilde\sigma^* G(X, Y)$$

$$= G(d\tilde\sigma(X), d\tilde\sigma(Y))$$

$$= G(d\sigma \circ d\varphi(X), d\sigma \circ d\varphi(Y))$$

$$= (\sigma^* G)(d\varphi(X), d\varphi(Y))$$

$$= (d\varphi(X))^T \underline{G} (d\varphi)(Y)$$

$$= X^T (d\varphi)^T \underline{G} (d\varphi) Y.$$

$\boxed{\text{First Fundamental Form}}$

Let $\sigma : U \twoheadrightarrow S \subseteq \mathbb{R}^m$ be a par. surface.

Defn 2 The first fundamental form G of σ is the $(0,2)$-tensor field on S obtained by restricting $\langle ?, ? \rangle$ to S i.e. at $u \in U$, it's the bilinear form on $T_u S$

$$G(X, Y) = \langle X, Y \rangle, \quad X, Y \in T_u S.$$

G is symmetric.

N.B. Smoothness of G clear from any e.g.

E.g. Consider surface of revolution

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))^T.$$

$$\partial_1 \sigma = (\dot{f}\cos v, \dot{f}\sin v, \dot{g})^T$$

$$\partial_2 \sigma = (-f\sin v, f\cos v, 0)^T$$

$$G_{11} = G(\partial_1\sigma, \partial_1\sigma) = \langle \partial_1\sigma, \partial_1\sigma \rangle$$

$$= \dot f(u)^2 + \dot g(u)^2$$

Sim. find coord. tensor

$$(G_{ij}) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} \dot f^2 + \dot g^2 & 0 \\ 0 & f^2 \end{pmatrix}$$

Note G symmetric $\Rightarrow$ $\underline{G}$ symm. matrix.

Also meridians $v = \text{const} \perp$ parallels $u = \text{const}$.

Curves on Surfaces

Defn 3 A parametrised curve on a surface $\sigma : U \to S$ is one of the form $\gamma = \sigma \circ c : I \to S$ where I is an interval & $c : I \to U$ is smooth.

Prop Writing $c(t) = (u^1(t), u^2(t))^T$,

$$\text{length } \gamma = \int_I \sqrt{(\sigma^* G)(\dot c(t), \dot c(t))}\, dt$$

$$\overset{\text{by multi-lin.}}{=} \int_I \sqrt{\sum_{i,j=1}^{2} G_{ij}(c(t))\, \dot u^i(t) \dot u^j(t)}\, dt$$

Proof $\text{Length} = \int \sqrt{G(\underbrace{\dot\gamma(t)}_{= d\sigma(\dot c(t))}, \dot\gamma(t))}\, dt$ $\square$

Upshot Geometry of distances on S can be computed on parameter space U, if you replace Euclid. metric $\langle\, ,\rangle$ with new $(0,2)$-tensor field $(\sigma^* G)(u)$.