Lecture 7: Surfaces

Aim Lecture Intro. 2-folds = surfaces

Surfaces

A parametrised surface is a par. 2-fold $\sigma : U \twoheadrightarrow S \subseteq \mathbb{R}^m$. Param. gives canonical ordered basis

$$\left\{ \frac{\partial \sigma}{\partial u^i} = \partial_i \sigma = (d\sigma)_u(e_i) \right\}_{i=1}^{2} \subset T_u S \text{ for each } u \in U.$$

Order is important for orientation of σ.

Warning re: Abused Notn @ Really should write $\frac{\partial \sigma}{\partial u^i}(u)$, $\partial_i \sigma(u)$ $\because$ they depend on $u \in U$.

(b) $\frac{\partial \sigma}{\partial u^i} = \partial_i \sigma$ can also denote the vector field $u \longmapsto (\partial_i \sigma)(u)$.

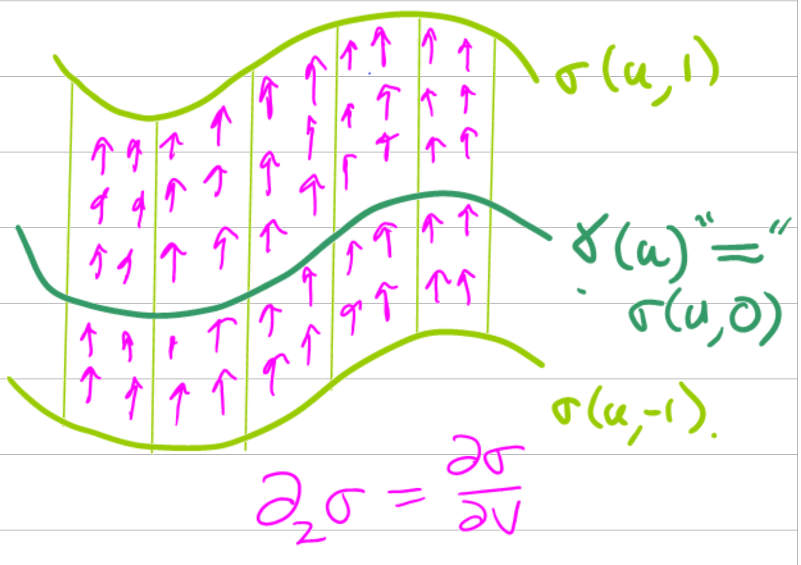
define later

Eg1 Cylinder

Param. curve $\gamma : I \to \mathbb{R}^2$, I open.

Define cylinder $\sigma : I \times \mathbb{R} \to \mathbb{R}^3$

$$(u,v) \mapsto \begin{pmatrix} \gamma(u) \\ v \end{pmatrix}$$



Vector field $\partial_2 \sigma = \frac{\partial \sigma}{\partial v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ at every $\sigma(u,v)$, $(u,v) \in I \times \mathbb{R}$.

Default for us so we'll omit this adjective

More gen.

Defn 1 A (smooth) vector field on σ or S is a fn of form
$$U \ni u \longmapsto \xi^1(u)\,\partial_1\sigma + \xi^2(u)\,\partial_2\sigma \in T_u S$$
where $\xi^1, \xi^2 \in C^\infty(U)$ i.e. co-ord. wrt $\partial_1\sigma, \partial_2\sigma$ are smooth fns of u.

Prop-Defn 1 A change of coordinates for σ is an open set $\tilde{U} \subseteq \mathbb{R}^2$ and a diffeo. $\varphi : \tilde{U} \xrightarrow{\sim} U$.

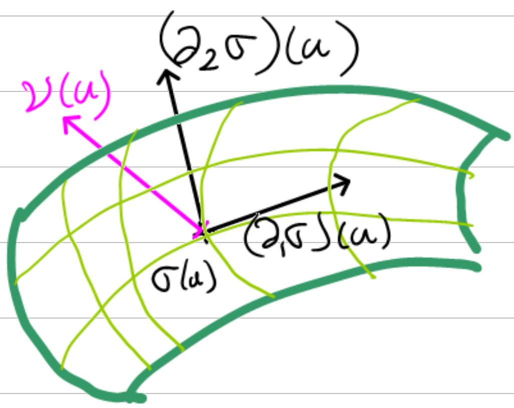
In this case, the reparametrisation $\tilde{\sigma} := \sigma \circ \varphi : \tilde{U} \twoheadrightarrow S \subseteq \mathbb{R}^m$ is also a par. surface. We say φ preserves (resp. reverses) orientation if $\det d\varphi > 0$ (resp. $\det d\varphi < 0$)

Proof Chain rule $\Rightarrow d\tilde{\sigma}_{\tilde{u}} = d\sigma_{\varphi(\tilde{u})} \circ d\varphi_{\tilde{u}}$ inj.

Also note $\det d\varphi$ can't change sign $\because \det (d\varphi)_u$ cont. in u & nonzero. $\square$

$\boxed{\text{Gauss Normal Vector}}$

Suppose par. surface $\sigma : U \xrightarrow{\sim} S \subset \mathbb{R}^3$ embedded in $\mathbb{R}^3$.



Defn 2 The Gauss normal to σ (or S) at $u \in U$ is
$$\nu(u) := \frac{\partial_1\sigma \times \partial_2\sigma}{|\partial_1\sigma \times \partial_2\sigma|} \in T_{\sigma(u)}\mathbb{R}^3$$
argument u dropped as per notn abuse.

Identifying $T_{\sigma(u)}\mathbb{R}^3 = \mathbb{R}^3$ get Gauss map
$$\nu : U \longrightarrow S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}.$$
$$u \longmapsto \nu(u)$$

Prop Let $\tilde\sigma : \tilde U \twoheadrightarrow S$ be reparam. wrt $\varphi : \tilde U \to U$. If $\tilde\nu$ is Gauss normal to $\tilde\sigma$ then $\tilde\nu(\tilde u) = \nu(\varphi(\tilde u))$ if φ preserves orient. & $\tilde\nu = -\nu$ if it reverses orient.

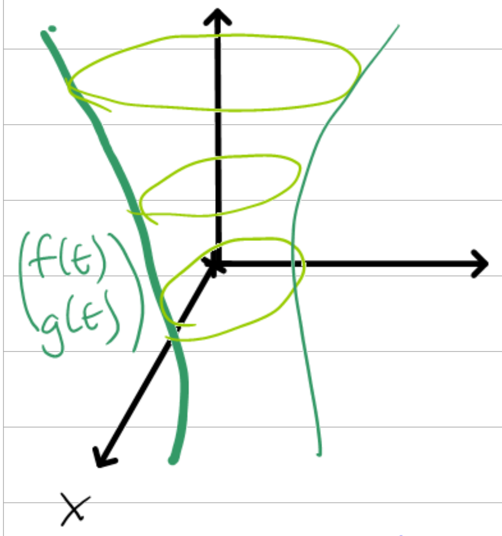
Proof: Let $p = \tilde\sigma(\tilde u) = \sigma(u)$ where $u = \varphi(\tilde u)$.
Both $\nu(u), \tilde\nu(\tilde u) \perp T_pS$ & unit length.
$\therefore$ Suff. check sign $\langle \nu, \partial_1\tilde\sigma \times \partial_2\tilde\sigma \rangle$.
Chain rule: $d\tilde\sigma_{\tilde u} \overset{*}{=} d\sigma_u \circ d\varphi_{\tilde u}$ so

Drop arguments $u, \tilde u$ below,
$$\langle \nu, \partial_1\tilde\sigma \times \partial_2\tilde\sigma \rangle = \det\left(\nu \;\middle|\; \overbrace{\partial_1\tilde\sigma \;\middle|\; \partial_2\tilde\sigma}^{d\tilde\sigma}\right)$$
$$\underset{\text{ex using } *}{=} \det\left(\nu \;\middle|\; \overbrace{\partial_1\sigma \;\middle|\; \partial_2\sigma}^{d\sigma}\right) \det\begin{pmatrix} 1 & 0 \quad 0 \\ \begin{matrix}0\\0\end{matrix} & d\varphi \end{pmatrix}$$
$$= \langle \nu, \partial_1\sigma \times \partial_2\sigma \rangle \det(d\varphi)$$
$> 0 \because \langle \nu, \nu \rangle = 1$ $\square$

Rem Hence, the orientation of surfaces in $\mathbb{R}^3$ correspond to "side" of surface ν points to.

Surfaces of Revolution

work in $\mathbb{R}^3_{x,y,z}$



Consider curve in xz-plane

$$I \longrightarrow \mathbb{R}^2_{x,z}$$
$$t \longmapsto \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

with $f(t) \neq 0$ for each t.

It generates the surface of revolution

$$\sigma : I \times \mathbb{R} \longrightarrow \mathbb{R}^3 : \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} f(u) \cos v \\ f(u) \sin v \\ g(u) \end{pmatrix}$$

CHECK $\partial_1 \sigma = \frac{\partial \sigma}{\partial u} = (\dot{f}(u) \cos v, \dot{f}(u) \sin v, \dot{g}(u))^T$

$$\partial_2 \sigma = \frac{\partial \sigma}{\partial v} = (-f(u) \sin v, f(u) \cos v, 0)^T.$$

Since $f(u) \neq 0$ & $(\dot{f}(u), \dot{g}(u)) \neq 0$,
$\partial_1 \sigma, \partial_2 \sigma$ lin. indep. $\Rightarrow$ σ par. surface.

$$\partial_1 \sigma \times \partial_2 \sigma = \begin{pmatrix} -f(u)\dot{g}(u) \cos v \\ -f(u) \dot{g}(u) \sin v \\ \dot{f}(u) f(u) \end{pmatrix}$$

If $f(u) > 0$ then

$$\nu(u,v) = \frac{1}{\sqrt{\dot{f}(u)^2 + \dot{g}(u)^2}} \begin{pmatrix} -\dot{g}(u) \cos v \\ -\dot{g}(u) \sin v \\ \dot{f}(u) \end{pmatrix}$$