# Lecture 6 : Tensors

**Aim Lecture** Intro. some multi-linear algebra

$\mathbb{F}$ = field, $V$ = fin. dim. $\mathbb{F}$-space.

i.e. vector spaces / $\mathbb{F}$.

## $\boxed{(0,s)\text{-tensors}}$

Let $V_1, \dots, V_s$ be vector spaces / $\mathbb{F}$

**Prop-Defn 1** A function $\varphi : V_1 \times \dots \times V_s \longrightarrow \mathbb{F}$ is linear in $V_i$ if for all $v_j \in V_j$, $j \neq i$, the function $\varphi(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_s)$ of $v$ is linear.

We say $\varphi$ is a multilinear form if it's linear in each $V_i$, $i = 1, \dots, s$.

The set of these is an $\mathbb{F}$-space with pointwise addn & scalar multn.

**Proof** Follows from closure axioms for linear maps. $\square$

**Eg. 1.** For matrix $A \in M_n(\mathbb{R})$, have multi-linear (= bilinear here) form $\langle ?, ? \rangle_A : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\langle x, y \rangle_A = x^T A y$.

**Defn 1** A $(0,s)$-tensor on $V$ is a multi-lin. form $\varphi : V^s \longrightarrow \mathbb{F}$. The vector space of these is denoted $V^{0,s}$.

**Eg. 2** $V^{0,1} = V^*$ the dual space of linear maps $V \to \mathbb{F}$.

Eg. 3 Let $v^*, w^* \in V$. Define $v^* \otimes w^* \in V^{0,2}$ by $(v^* \otimes w^*)(v,w) = v^*(v)\, w^*(w)$

ex. CHECK multi-lin.

$\boxed{\text{Co-ordinates}}$ Intro. coordinates to compute with (0,2)-tensors. Need basis.

Consider basis $\{\varepsilon_1, \dots, \varepsilon_n\} \subset V$ & dual basis $\{\varepsilon^1, \dots, \varepsilon^n\} \subset V^*$ ie $\varepsilon^i$ is unique linear map s.t $\varepsilon^i(\varepsilon_j) = \delta^i_j = \begin{cases} 1 & i=j \\ 0 & i \neq j. \end{cases}$

Hopefully you know this is a basis for $V^*$.

Prop-Defn 2 a $(\varepsilon^i \otimes \varepsilon^j)(\varepsilon_r, \varepsilon_s) = \begin{cases} 1 & i=r, j=s \\ 0 & \text{else}. \end{cases}$

b $V^{0,2}$ has dim $n^2$ with induced basis $\{\varepsilon^i \otimes \varepsilon^j \mid i,j = 1, \dots, n\}$.

c If $g \in V^{0,2}$ then $g = \sum g_{ij}\, \varepsilon^i \otimes \varepsilon^j$ where $g_{ij} = g(\varepsilon_i, \varepsilon_j)$.

The coordinate tensor of $g$ wrt $\varepsilon_1, \dots, \varepsilon_n$ is $(g_{ij})$. ie. coords wrt induced basis.

Proof a $(\varepsilon^i \otimes \varepsilon^j)(\varepsilon_r, \varepsilon_s) = \varepsilon^i(\varepsilon_r)\ \varepsilon^j(\varepsilon_s) = \delta^i_r \delta^j_s$

c Let $v = \sum v^i \varepsilon_i$ $w = \sum w^j \varepsilon_j$, $\tilde{g} = \sum g_{ij} \varepsilon^i \otimes \varepsilon^j$ as above.

$$g(v,w) = g\left(\sum v^i \varepsilon_i, \sum w^j \varepsilon_j\right)$$
$$= \sum v^i g\left(\varepsilon_i, \sum w^j \varepsilon_j\right) = \sum v^i w^j g(\varepsilon_i, \varepsilon_j)$$
$$= \sum g_{ij} v^i w^j = \sum g_{ij} \varepsilon^i(v) \varepsilon^j(w)$$
$$= \tilde{g}(v,w)$$

b Note $\{\varepsilon^i \otimes \varepsilon^j\}$ lin. indep. by ⓐ & spans by ⓒ. □

Proof ⟹

**Formula 1** $g\left(\sum v^i \varepsilon_i, \sum w^j \varepsilon_j\right) = \sum g_{ij} v^i w^j$.

$\boxed{(1,1)\text{-tensors}}$ We make the unconventional

**Defn 2** A (1,1)-tensor on $V$ is a linear map $L : V \longrightarrow V$. The vector space of these is denoted $V^{1,1} = \operatorname{End} V$.

**Rem** One can more gen. define $(r,s)$-tensors.

These are related to $(0,2)$-tensors by

**Prop-Defn 3** Let $G \in V^{0,2}$, $L \in V^{1,1}$. Get a new $(0,2)$-tensor $G.L \in V^{0,2}$ defined by
$$(G.L)(v,w) := G(v, Lw)$$

**Proof** easy ex. □

**Eg 4** If $(\mathbb{R}^n)^{0,2} \ni G = \langle ?, ? \rangle$ & $A \in M_n(\mathbb{R})$ then $G.A = \langle ?, ? \rangle_A$.

Let $B = \{\varepsilon_1, \dots, \varepsilon_n\} \subset V$ be a basis, $G \in V^{0,2}$, $L \in V^{1,1}$. Let matrix repr. $L$ wrt $B$ be $\underline{L} = (L^i_j)$ with $i$-th row $j$-th entry $L^i_j$ so $L(\sum_j v^j \varepsilon_j) = \sum_{i,j} L^i_j v^j \varepsilon_i$ whose co-ord. vector is $\underline{L}\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$.

If coord tensor of $G$ is $(G_{ij})$, define associated matrix $\underline{G}$ wrt $B$ by: $\underline{G}^i_j = G_{ij}$.

The co-ord. tensor of $G.L$ wrt $B$ is

$$(G.L)_{ij} = (G.L)(\varepsilon_i, \varepsilon_j) = G(\varepsilon_i, \sum L^r_j \varepsilon_r)$$
$$= \sum_r G_{ir} L^r_j$$

Formula 2 $(G.L)_{ij} = \sum G_{ir} L^r_j = \underline{G}^i_r L^r_j$

or equiv.

Formula 2' $\underline{G.L} = \underline{G}\ \underline{L}$ (matrix product).

Rem. Can also define $L.G \in V^{0,2}$ by $(L.G)(v,w) = G(Lv, w)$.

Prop Let $[v]_B \in \mathbb{R}^n$ denote co-ords. of $v \in V$ wrt. $B$. Then

$$G(v,w) = [v]_B^T \underline{G} [w]_B = \langle [v]_B, [w]_B \rangle_{\underline{G}}$$

Proof This just restates Formula 1:

$$G(v,w) = \sum G_{ij} v^i w^j = \sum_i v^i (\sum \underline{G}^i_j w^j)$$
$$= \langle [v]_B, \underline{G}[w]_B \rangle \ \square$$