Lecture 5: Hopf's Umlaufsatz

Aim Lecture Compute rotn no. for simple closed curves. This illustrates central theme of course

$$\int_M \text{nice local inv. } dM = \text{global top. inv. of } M$$

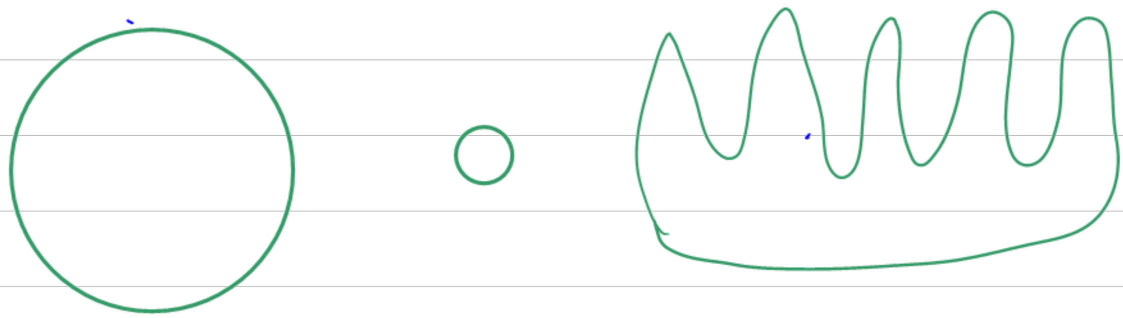
Umlaufsatz

Thm (Hopf) Let $\gamma : [a,b] \longrightarrow C \subset \mathbb{R}^2$ be a p-wise sm. simple closed curve with ext. angles $\in (-\pi, \pi)$. Then the rotn no $n_\gamma = \pm 1$. (Sign is + if you go around ↺)

In particular, if γ is unit speed smooth,

$$\frac{1}{2\pi}\int_{[a,b]} \kappa_s(u)\, du = \pm 1$$

Eg



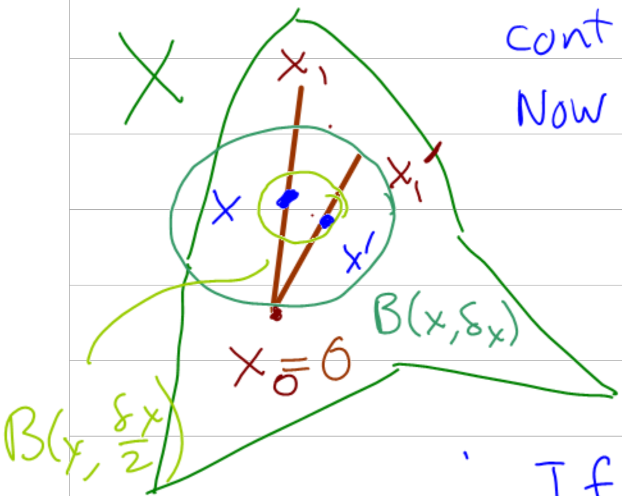
Preliminaries

Lemma Let $X \subseteq \mathbb{R}^2$ be star-shaped wrt. $x_0 \in X$ i.e. $x \in X \Rightarrow$ line segment $\overline{xx_0} \subseteq X$. Any cont. $f : X \longrightarrow S^1 \subset \mathbb{R}^2$ has an ang. fn θ.

Proof (FYI not exam)

Start by picking $\theta(x_0)$ s.t. $f(x_0) = \begin{pmatrix} \cos\theta(x_0) \\ \sin\theta(x_0) \end{pmatrix}$

Lemma lect. 3 + addendum defines $\theta : X \to \mathbb{R}$, angular on any ray $\overline{x_0 x_1}$, $x_1 \in X$.



Fix x_1 & $\varepsilon \in (0, \pi)$. We prove θ cont. at x_1.

Now f cont. $\Rightarrow$ for each $x \in \overline{x_0 x_1}$, $\exists\, \delta_x > 0$ s.t.
$|x' - x| < \delta_x \Rightarrow$
$|f(x') - f(x)| < \varepsilon/2$

$\therefore$ If $x'' \in \overline{x_0 x_1} \cap B(x, \delta_x/2)$

openball centre x radius $\delta_x/2$

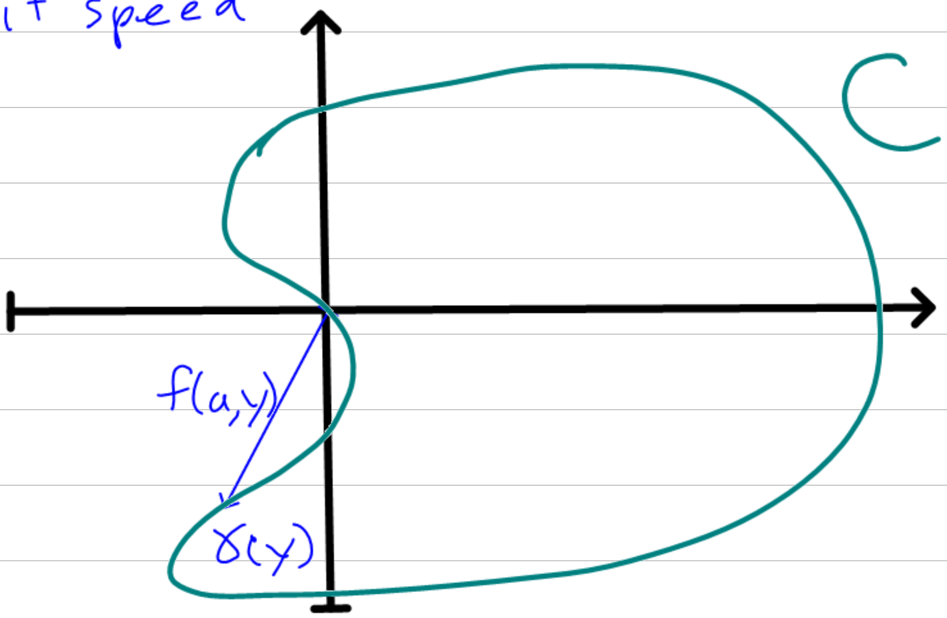
& $|x' - x''| < \delta_x/2 \Rightarrow$
$|f(x') - f(x'')| < |f(x') - f(x)| + |f(x) - f(x'')|$
$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$

Covering $\overline{x_0 x_1}$ with $B(x, \delta_x/2)$, $x \in \overline{x_0 x_1}$ & using compactness $\Rightarrow \exists\, \delta > 0$ s.t. for any $x \in \overline{x_0 x_1}$, $x' \in X$ with $|x' - x| < \delta \Rightarrow \theta(x') - \theta(x) = 2\pi k(x', x) + \varepsilon'$ where $|\varepsilon'| < \varepsilon$, $k(x', x) \in \mathbb{Z}$.

Suff. show $k(x_1', x_1) = 0$ for $|x_1' - x_1| < \delta$. WLOG, $x_0 = 0$. But cont. fn of $t \in [0,1]$
$\nu(t) = \theta(t x_1') - \theta(t x_1') = 2\pi\, k(t x_1', t x_1) + \varepsilon'(t)$
$\in \mathbb{R} - \{\pm \pi\}$
has value 0 at $t = 0$ so image of ν lies in $(-\pi, \pi)$ & $k(x_1', x_1) = 0$. $\square$

Proof THM $\gamma : [a,b] \to C \subset \mathbb{R}^2_{x,y}$

Rotating & translating C, can ass. unit speed



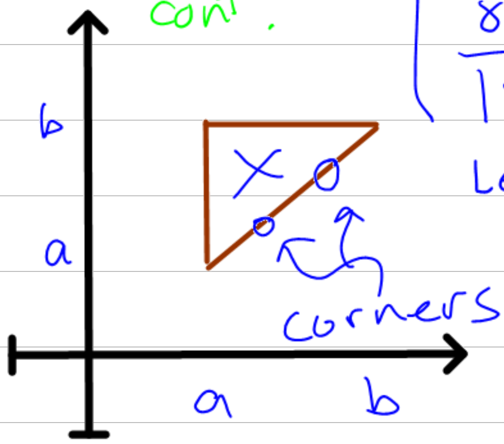
& $\gamma(a) = \gamma(b) = 0$ but $C \cap$ neg. y-axis $= \emptyset$ & corners $a_i \in (a,b)$. Let

$X = \{(x,y) \mid a \le x \le y \le b$ except $x = y = $ some $a_i\}$

define cont. $f : X \to S^1 \subset \mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \dot{\gamma}(x) & , x = y \\ -\dot{\gamma}(a) & , (x,y) = (a,b) \\ \dfrac{\gamma(y)-\gamma(x)}{|\gamma(y)-\gamma(x)|} & , \text{else} \end{cases}$$

ex. Check f cont.



Lemma $\Rightarrow$ f has ang. fn say $\theta : X \to \mathbb{R}$ & can ass. $\theta(a,a) \in (-\pi, \pi)$

Claim 1 $\theta(b,b) - \theta(a,a) = \pm 2\pi$.

$\because$ $\theta(a,a) \longleftrightarrow \dot{\gamma}(a)$, $\theta(a,b) \longleftrightarrow -\dot{\gamma}(a)$

$\Rightarrow$ $\theta(a,b) - \theta(a,a) \equiv \pi \mod 2\pi$.

But $\theta(a,t) \in (-\pi, \pi)\ \forall t \in (a,b)$

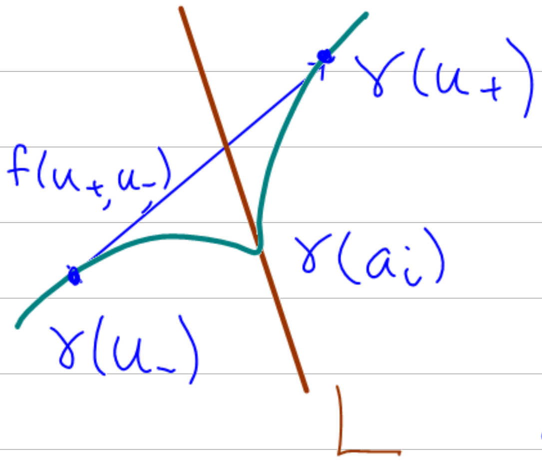
so $\theta(a,b) - \theta(a,a) = \pm\pi$ Can check same sign.

Sim. $\theta(b,b) - \theta(a,b) = \pm\pi$. $\square$

Rem: Note sign + if ↺ ie. keep inside on LHS.

Claim 2 $\theta(a_i^+, a_i^+) - \theta(a_i^-, a_i^-) = \text{ext. angle } \Delta_i\theta$

∵



Note

LHS $= \Delta_i\theta + 2k\pi$, $k \in \mathbb{Z}$,

Suff. show $k = 0$.

$\Delta_i\theta \neq \pi \Rightarrow \exists\ \infty$ no lines L through $\gamma(a_i)$ s.t. for $u_- < a_i < u_+$ close enough, $\gamma(u_\pm)$ are on either side of L.

∴ $f(u_+, u_-) \not\parallel L$ so $\exists$ closed interval $I \subseteq \mathbb{R}$ of length $< \pi$ s.t. $\theta(u_+, u_-) \in I$.

∴ $\theta(a_i^+, a_i^+) = \theta(a_i, a_i^+) \in I$

& sim. $\theta(a_i^-, a_i^-) \in I$.

∴ $\theta(a_i^+, a_i^+) - \theta(a_i^-, a_i^-) \in (-\pi, \pi)$

$\Rightarrow k = 0$. $\square$

Claim 3 For consecutive corners a_i, a_{i+1},

$$\theta(a_{i+1}^-, a_{i+1}^-) - \theta(a_i^+, a_i^+) = \int_{a_i}^{a_{i+1}} \kappa_s(u)\,|\dot{\gamma}(u)|\,du$$

by Prop Lect. 3. $\square$

Claims 1-3 $\Rightarrow$ Thm. $\square$