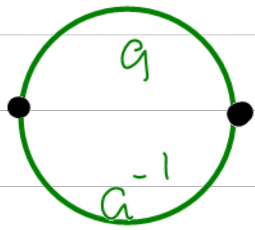
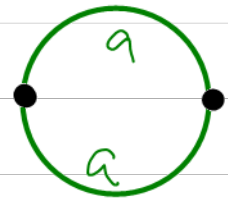
# Lecture 30: Classification of Surfaces II

Aim Complete classification.

We consider a cell complex $K$ which is a symbol $a_1 \dots a_n$. Only 2-gons are




$S^2$




$\mathbb{P}^2(\mathbb{R})$.

So assume $n > 2$ & no consec. $aa^{-1}$ in symbol (Lemma 2 Lect. 29).

## Handles

**Defn 1** A handle is an occurrence of $aba^{-1}b^{-1}$ in a symbol.

**Formula 1** For edge $a$, edge seq. $\alpha, \beta, \gamma$

$$a\alpha\beta a^{-1}\gamma \sim a\beta\alpha a^{-1}\gamma$$ $a$'s on left & right different.

**Proof** $a\alpha\beta a^{-1}\gamma \sim a\alpha b,\ b^{-1}\beta a^{-1}\gamma$ new edge $b$, $= \gamma b^{-1}\beta a^{-1}$

$\sim \gamma b^{-1}\beta\alpha b \sim b^{-1}\beta\alpha b\gamma$

$\sim a\beta\alpha a^{-1}\gamma$ relabelling □

**Formula 1′** If $\gamma = \gamma'\gamma''$, then $\gamma'' a\alpha\beta a^{-1}\gamma' \sim \gamma'' a\beta\alpha a^{-1}\gamma'$

**Prop 1** A cross-cap and handle can be replaced by 3 cross-caps.

**Proof** $aa\alpha bcb^{-1}c^{-1}\beta$

$\sim a b^{-1} c^{-1} \beta a c^{-1} b^{-1} \alpha^{-1}$ by Formula Lect. 29

$= b^{-1} c^{-1} \beta a c^{-1} b^{-1} \alpha^{-1} a$

$\sim b^{-1} b^{-1} a^{-1} \alpha c^{-1} \beta a c^{-1}$ form. Lect. 29

$= c^{-1} \beta a c^{-1} b^{-1} b^{-1} a^{-1} \alpha$

$\sim c^{-1} c^{-1} \alpha^{-1} a b b \beta a$ formula Lect 29

$= a b b \beta a c^{-1} c^{-1} \alpha^{-1}$

$\sim a a \alpha c c b b \beta$ form. Lect 29. $\square$

Formula 2 $a \alpha b \beta a^{-1} \gamma b^{-1} \delta \sim a b a^{-1} b^{-1} \delta \gamma \beta \alpha$

Proof Use formula 1′ repeatedly to see

$a \alpha b \beta a^{-1} \gamma b^{-1} \delta \sim a b \beta \alpha a^{-1} \gamma b^{-1} \delta$

$\sim a b a^{-1} \gamma \beta \alpha b^{-1} \delta$

$= a^{-1} \gamma \beta \alpha b^{-1} \delta a b$

$\sim a^{-1} b^{-1} \delta \gamma \beta \alpha a b = a b a^{-1} b^{-1} \delta \gamma \beta \alpha$

$\square$

$\boxed{\text{Proof of Classification}}$

Thm Any triangulable surface $S$ is homeo to one of the following

a A sphere with $g \geq 0$ handles $|K|$ where $K$ has symbol $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$ & by default, $K$ is $a a^{-1}$ when $g = 0$ ie $|K| \cong S^2$.

b A surface $|K|$ with $k$ cross-caps where $K$ is symbol $a_1 a_1 a_2 a_2 \ldots a_k a_k$.

Proof We may suppose $S = |K|$ &, by Lect. 29, lemma 1, $K$ is a symbol $a_1 \ldots a_n$.

The only 2-gons give the sphere $|aa^{-1}|$ and $\mathbb{P}^2(\mathbb{R}) = |aa|$ so suppose $n \geq 3$. Hence can ass. by Lect. 29 Lemma 3, $K$ has only 1 vertex.

Note formation of cross-caps (Lect. 29, Formula) & handles (Formula 2) & replacing crosscap + handle with 3 cross-caps only use cut (SF) & paste (GF) so leave $K$ with 1 vertex.

Non-orient. case Prop 1, Lect 29 ⇒ can ass. all double occurrence of edges are cross-caps & ass at least one. If symbol has pattern $a\alpha b\beta a^{-1}\gamma b^{-1}\delta$, can form handle without changing cross-caps & then replace with cross-caps. Need show only cross-caps left so ass. instead symbol has form $a\alpha a^{-1}\beta$ & no edge or its inverse in $\alpha$ appears in $\beta$. This case finished by

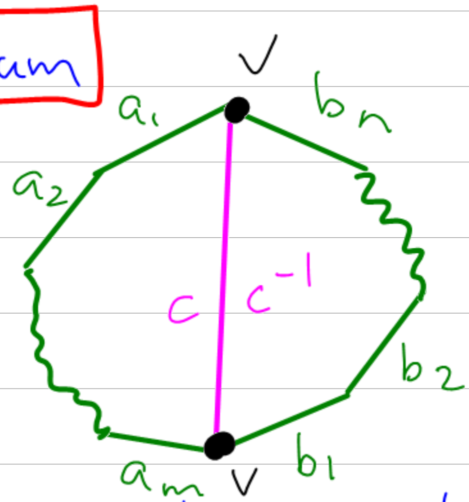
**Lemma** Given symbol $a\alpha a^{-1}\beta$ where no edge of $\alpha$ or its inverse appears in $\beta$, then the head & tail of $a$ are distinct vertices.

**Proof by Picture**


Contradicts only 1 vertex. $\square$

**Orientable Case** Suppose we've shown inductively symbol has form $\eta_1 \ldots \eta_k \alpha$ where $\eta_i$ are handles. Lemma $\Rightarrow \alpha = \beta a \gamma b \delta a^{-1} \varepsilon b^{-1}$.
Formula 2 $\Rightarrow$ symbol $a\gamma b \delta a^{-1} \varepsilon b^{-1} \eta_1 \ldots \eta_k \beta$ $\sim a b a^{-1} b^{-1} \eta_1 \ldots \eta_k \beta \varepsilon \delta \gamma$. Done by induction. $\square$

## Connected Sum

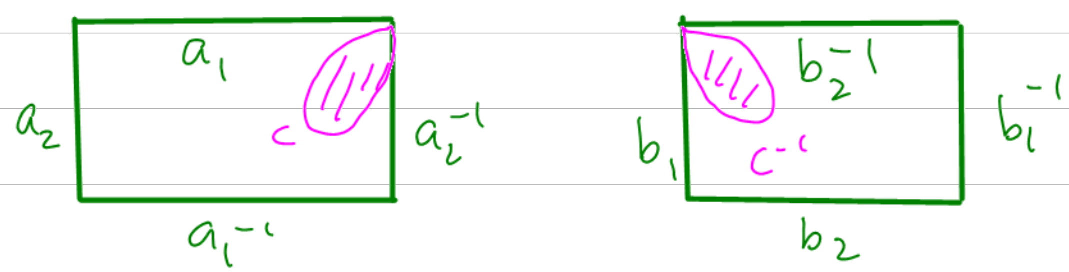


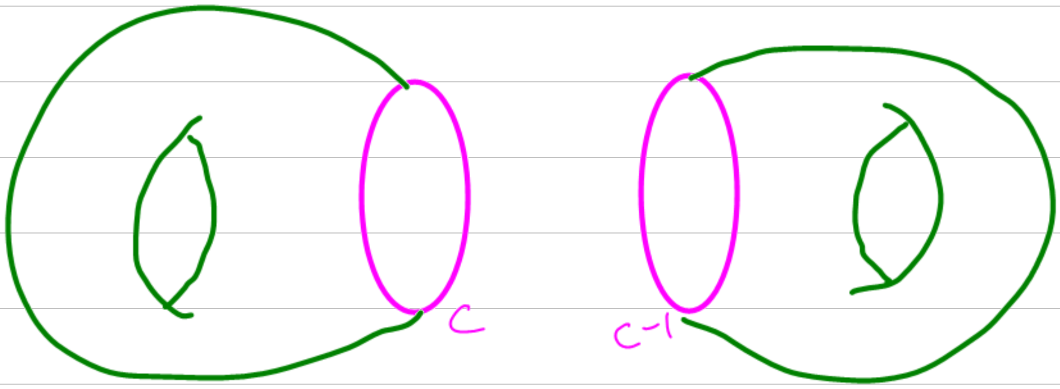

Consider symbol $a_1 \ldots a_m b_1 \ldots b_n$ where no elt of $\{a_i\}$ or its inverse is in $\{b_i\}$ & vice versa. Suppose tail $a_1 = v =$ head $a_m$.

**Defn 2** We say $X = |a_1 \ldots a_m b_1 \ldots b_n|$ is the **connected sum** of $X_a = |a_1 \ldots a_m|$ and $X_b = |b_1 \ldots b_n|$ & write $X = X_a \# X_b$.

$X$ is obtained by deleting small discs from $X_a, X_b$ with boundaries $c, c^{-1}$ & gluing these together.

**Eg.** $X_a = \mathbb{I}^2$ $\qquad$ $X_b = \mathbb{I}^2$






one handle　　　another handle.