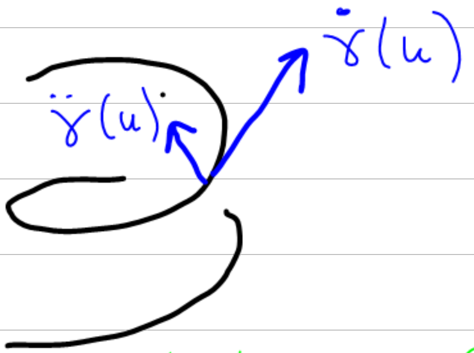
Lecture 3: Curvature

Aim Lecture Study how curves curve.

Osculating Frame



Here we can view $\dot{\gamma}$ as a fn $\dot{\gamma} : I \to \mathbb{R}^m$ & differentiate again to get $\ddot{\gamma} : I \to \mathbb{R}^m$.

Note identification $T_{\gamma(u)}\mathbb{R}^m = \mathbb{R}^m$ here.

Defn 1 A par. curve $\gamma : I \longrightarrow \mathbb{R}^m$ is **osculating** if $\forall u \in I$, $\dot{\gamma}(u), \ddot{\gamma}(u) \in \mathbb{R}^m$ are lin. indep.

The **osculating plane** at u is $\mathrm{Span}(\dot{\gamma}(u), \ddot{\gamma}(u)) \leq T_{\gamma(u)}\mathbb{R}^m$ or $\gamma(u) + \mathrm{Span}(\dot{\gamma}(u), \ddot{\gamma}(u)) \subseteq \mathbb{R}^m$.

Eg. The line $\gamma(u) = u\begin{pmatrix}1\\2\\3\end{pmatrix}, u \in \mathbb{R}$ is not osculating.

The **osculating frame** of an osculating curve γ is for each $u \in I$, the o/n set $\varepsilon_1(u), \varepsilon_2(u) \in T_{\gamma(u)}\mathbb{R}^m$ defined by:

$$\varepsilon_1(u) = \dot{\gamma}(u) / |\dot{\gamma}(u)|$$

$$\varepsilon_2(u) \in \mathrm{Span}(\dot{\gamma}(u), \ddot{\gamma}(u))$$

is s.t. $\varepsilon_2(u) \perp \varepsilon_1(u)$, $|\varepsilon_2(u)| = 1$ and $\langle \ddot{\gamma}(u), \varepsilon_2(u) \rangle > 0$.

Curvature

Let $\gamma : I \longrightarrow \mathbb{R}^m$ be an osculating curve.

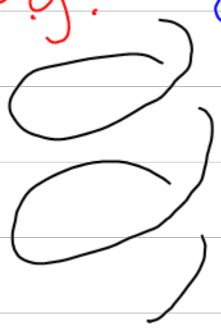
Prop-Defn 1 The curvature of γ is
$$\kappa(u) = \frac{1}{|\dot\gamma(u)|} \langle \dot\varepsilon_1(u), \varepsilon_2(u) \rangle .$$

It is independent of parametrisation
i.e. for change of var. $\phi : \tilde I \to I$, $\tilde\gamma = \gamma \circ \phi$
$\tilde\kappa(\tilde u) = \kappa(\phi(\tilde u))$ where $\tilde u \in \tilde I$ &
$\tilde\kappa$ is curvature of $\tilde\gamma$.

Proof Calculating $\dot{\tilde\gamma}$, $\ddot{\tilde\gamma}$ using chain rule shows (see PS) osculating frame of $\tilde\gamma$ is
$$\tilde\varepsilon_i(\tilde u) = \varepsilon_i(\phi(\tilde u)) \text{ if } \phi \text{ preserves orient.}$$
Let $u = \phi(\tilde u)$.
$$\tilde\kappa(\tilde u) = \frac{1}{|\dot{\tilde\gamma}(\tilde u)|} \langle \dot{\tilde\varepsilon}_1(\tilde u), \tilde\varepsilon_2(\tilde u) \rangle$$
$$= \frac{1}{\dot\phi(\tilde u) |\dot\gamma(u)|} \langle \dot\phi(\tilde u) \dot\varepsilon_1(u), \varepsilon_2(u) \rangle$$
$$= \kappa(u) \quad \because \dot\phi(\tilde u) > 0 .$$

ex. prove orient. reversing case $\square$

Eg. $\gamma(u) = (\cos u, \sin u, u)^T$



$$\dot\gamma(u) = (-\sin u, \cos u, 1)^T$$
$$|\dot\gamma(u)| = \sqrt{2} \quad \text{so}$$
$$\varepsilon_1(u) = \tfrac{1}{\sqrt2} (-\sin u, \cos u, 1)^T .$$

$\ddot{\gamma}(u) = (-\cos u, -\sin u, 0) = \varepsilon_2(u)$

$\because \langle \varepsilon_2(u), \varepsilon_1(u)\rangle = 0$ & $|\varepsilon_2(u)| = 1$.

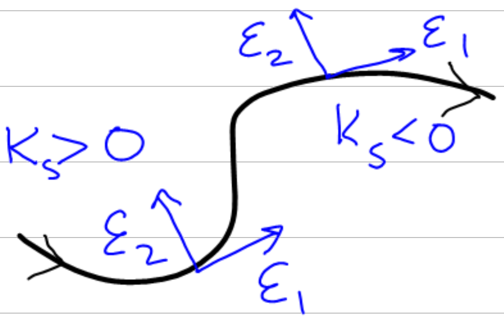
$$K(u) = \frac{1}{|\dot{\gamma}(u)|}\langle \dot{\varepsilon}_1(u), \varepsilon_2(u)\rangle = \frac{1}{2}$$

Rem If γ doesn't osculate at u i.e. $\dot{\gamma}(u) \parallel \ddot{\gamma}(u)$, define curvature $K(u) = 0$.

Signed Curvature

for curves on oriented surfaces

Let $\gamma: I \to \mathbb{R}^2$ be a plane curve i.e. in $\mathbb{R}^2$.



Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be rotn $90°$ ↺ in $\mathbb{R}^2$.

Prop-Defn 2 The Frenet frame of a plane curve γ is $\varepsilon_1(u) = \frac{1}{|\dot{\gamma}(u)|}\dot{\gamma}(u)$, $\varepsilon_2(u) = J\varepsilon_1(u)$.

It's signed curvature is $K_s(u) = \frac{1}{|\dot{\gamma}(u)|}\langle \dot{\varepsilon}_1(u), \varepsilon_2(u)\rangle$.

It's independent of orient. preserving change of var.

Proof ex.

Rem. $K_s = \pm K$

Let $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$.

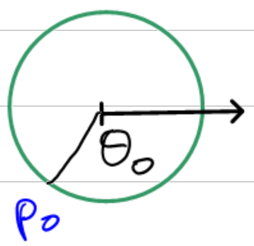
Defn 2 Let $f: X \to S^1$ be a continuous function. An angular function for

f is a cont. fn $\theta : X \to \mathbb{R}$ s.t.

$$f(x) = \begin{pmatrix} \cos\theta(x) \\ \sin\theta(x) \end{pmatrix} \quad \forall x \in X.$$

Lemma Let $I \subseteq \mathbb{R}_u$ be an interval & $e : I \to S^1$ be cont. Then there exists an angular fn θ for e.

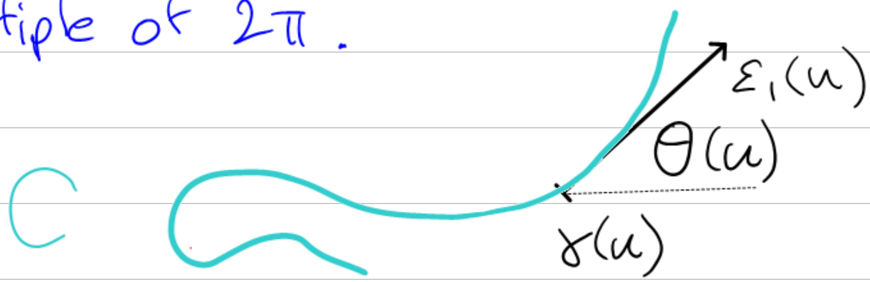
Proof see PS. But note for any $\theta_0 \in \mathbb{R}$, $p_0 = (\cos\theta_0, \sin\theta_0)^T$ have homeo



$$h : (\theta_0, \theta_0 + 2\pi) \longrightarrow S^1 - p_0$$
$$\theta \longmapsto (\cos\theta, \sin\theta)^T$$

so if $\mathrm{im}\, e \subseteq S^1 - p_0$ then can pick ang. fn $\theta = h^{-1} \circ e : I \to S^1 - p \to (\theta_0, \theta_0 + 2\pi)$

Addendum Furthermore, θ unique up to adding integer multiple of 2π. $\square$



Prop (Turning tangents) For curve $\gamma : I_u \to \mathbb{R}^2$, θ ang. fn for $\varepsilon_1 = \dot\gamma / |\dot\gamma|$ we have

$$\kappa_s(u) = \dot\theta(u) / |\dot\gamma(u)|.$$

Proof: $\varepsilon_1(u) = \begin{pmatrix} \cos\theta(u) \\ \sin\theta(u) \end{pmatrix} \Rightarrow \dot\varepsilon_1(u) = \dot\theta(u) \begin{pmatrix} -\sin\theta(u) \\ \cos\theta(u) \end{pmatrix}$

$$= \dot\theta(u)\, \varepsilon_2(u).$$

$$\kappa_s(u) = \frac{1}{|\dot\gamma(u)|} \langle \dot\varepsilon_1, \varepsilon_2 \rangle = \dot\theta(u) / |\dot\gamma(u)|. \quad \square$$