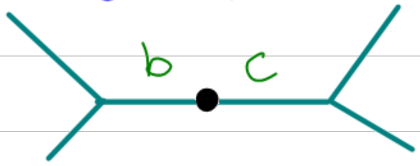
Lecture 29: Classification of Surfaces I

Aim Begin classifying triang. surface $|K|$.

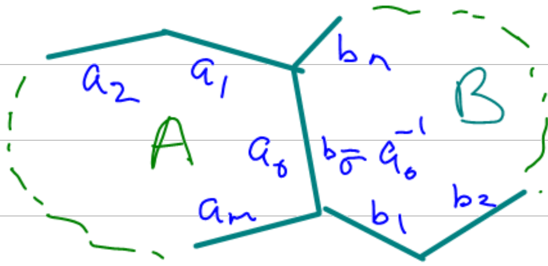
Elementary Gluing

(GE) We can perform inverse of (SE) subdividing edge of K as follows.



Suppose $\{b, c^{-1}\}$ is a vertex so bc & $c^{-1}b^{-1}$ occur in boundaries of 2 faces. We glue edges (GE) replacing b, c with single new edge a.

(GF) To invert subdividing faces (SF)



$\partial A = a_1 \ldots a_m a_0$, $\partial B = a_0^{-1} b_1 \ldots b_n$

If faces $B \neq A^{\pm 1}$ have common boundary edge $b_0 = a_0^{-1}$ as in picture we can glue faces (GF) & replace A B with C where $\partial C = a_1 \ldots a_m b_1 \ldots b_n$, elim. $a_0^{\pm 1}$.

Reduction to Single Face & Vertex

Lemma 1 A cell complex K is homeo. to

one consisting of a single face pair.

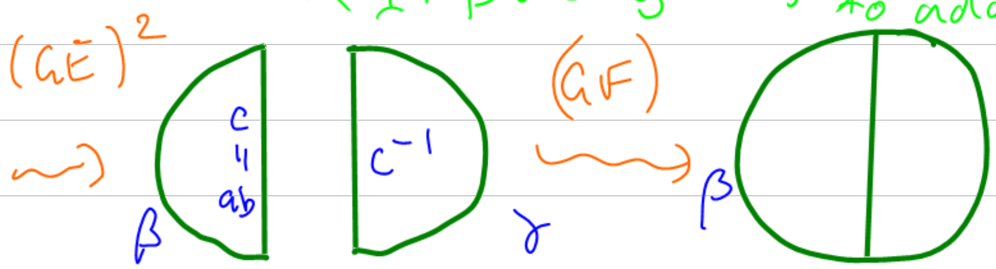
Proof If K has more than 1 face pair, then K conn. $\Rightarrow$ there are faces $A \neq B^{\pm 1}$ with common boundary edge. Apply (GF) to reduce no. faces & repeat until 1 face pair left. $\square$

Def^n 1 Ass. now K has 1 face pair $A^{\pm 1}$, so determined by one of its boundaries, say $a_1 \dots a_m$. This boundary called a symbol as is K. Abuse not^n: $K = a_1 \dots a_m$

Lemma 2 In any symbol $a_1 \dots a_n$, $n > 2$, elim. consecutive aa^{-1} gives homeo. symbol.

Proof Apply following (SF), $(GE)^2$, (GF).

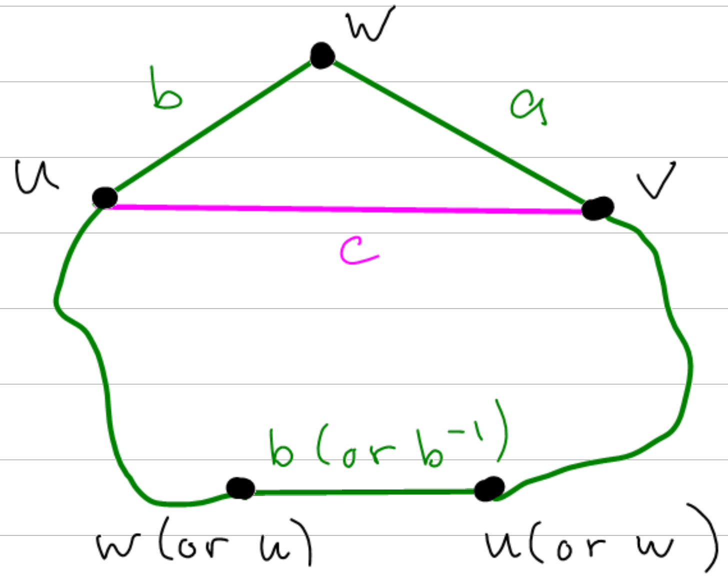
↖ If βγ single edge, need to (SE) to add b, then



(GE) to get rid of added vertex. $\square$

Lemma 3 A symbol is homeo. to a 2-gon or one with only 1 vertex.

Proof If not, $\exists$ edge a with distinct head & tail w, v. Lemma 2 $\Rightarrow$ can ass. picture below with $b \neq a^{-1}$.



Cutting (SF) along C & pasting (GF) abc back along b produces new symbol where v is the tail of 1 more edge than previously. Repeat until all vertices are v.

$\square$

$\boxed{\text{Cross-caps}}$

Below we let $\alpha, \beta, \gamma, \delta$ denote sequences of edges in a symbol/boundary.

Defn 2 If aa occurs in a symbol, it's called a cross-cap.

Rem By defn. of cell complex, if an edge a occurs in a symbol, then $a^{\pm 1}$ also occurs elsewhere & there is no other occurrence of $a^{\pm 1}$.

Prop 1 Any symbol is homeo. to one s.t. any edge a occurring twice appears as a cross-cap aa.

Proof Follows immediately from

Formula The symbol $a\alpha a\beta \sim aa\beta^{-1}\alpha$ where a is an edge, & $\sim$ means corresp. cell complexes are homeo.

Proof (SF) shows $a\alpha a\beta$ homeo to cell complex with two face pairs

$$a\alpha b,\ b^{-1}a\beta = a\beta b^{-1} = (b\beta^{-1}a^{-1})^{-1}.$$

(GF) $\Longrightarrow$

$$a\alpha a\beta \sim b\beta^{-1}\alpha b \sim bb\beta^{-1}\alpha \sim aa\beta^{-1}\alpha$$

relabel b with a $\square$

$\boxed{\text{Real Projective Plane}}$

$x \sim -x$

Prop 2 The geom. realisation $|aa|$ is homeo. to $\mathbb{P}^2(\mathbb{R})$.

Proof Let Δ = closed unit disc in $\mathbb{R}^2$ so

$|aa| = \Delta/\sim$ where $x \sim -x$ for $x \in \partial\Delta$

Consider homeo $g: [0,1] \to [0,\infty] : r \mapsto \tan\frac{\pi r}{2}$

Have cont.

$$f(r\cos\theta, r\sin\theta) = \begin{cases} (g(r)\cos\theta : g(r)\sin\theta : 1) & r<1 \\ (\cos\theta : \sin\theta : g(r)^{-1}) & r>0 \end{cases}$$

$\because$ for $0<r<1$, two right hand expressions agree & each is continuous.

Also f identifies $\pm(\cos\theta, \sin\theta)$ so univ. property of quotients gives cont $\bar{f}: |aa| \to \mathbb{P}^2(\mathbb{R})$.

Also $\bar{f}|_{\Delta^{int}}$ is a bij$^{\underline{n}}$ from Δ^{int} onto the "patch" $V_3 = \{(x:y:z) \mid z \neq 0\}$ whilst $\bar{f}|_{\partial\Delta}$ is a bij$^{\underline{n}}$ onto the line at ∞.

$\therefore$ $\bar{f}$ is a cont. bij$^{\underline{n}}$ from compact to Hausdorff so is an homeo. $\square$