Lecture 28: Theory of Surfaces

Aim Summarise main thms of surface theory.

Topological Invariance

Thm 1 Let K, K' be cell complexes with $|K|, |K'|$ homeo. Then K, K' are homeo.

Proof Lect. 31-33.

Defn 1 Let S be a compact top. surface & $\varphi: |K| \xrightarrow{\sim} S$ an homeo. We define the **Euler characteristic** of S to be $\chi(S) = \chi(K)$. We say S is **orientable** if K is.

This is well-defined $\because$ given another $\varphi': |K'| \xrightarrow{\sim} S$, thm 1 $\Rightarrow$ K, K' homeo.

Cor 1 If S, S' are homeo. compact surfaces with S triangulable then so is S' & $\chi(S) = \chi(S')$ & S is orientable iff S' is.

Proof Consider homeo $f: S \to S'$ & triang. $\varphi: |K| \xrightarrow{\sim} S$. Then $f\varphi: |K| \to S'$ is also a triang. so can use K in both cases to determine χ & orientability. $\square$

Eg 1 $\chi(S^2) = 2$, $\chi(\mathbb{T}^2) = 0 \Rightarrow S^2, \mathbb{T}^2$ not homeo.!

N.B. This is a remarkable result as it's hard to show $\nexists$ homeo. $S^2 \simeq \mathbb{T}^2$.

$\mathbb{T}^2$ orient. but Klein bottle not $\Rightarrow$ not homeo..

Existence of Triangulations

Thm 2 * Any conn. compact top. surface S has a triangulation $\varphi : |K| \longrightarrow S$

* For any given open cover $S = \bigcup V_\alpha$, we may assume K is suff. refined that every face $\Delta_A \subseteq |K|$, $A \in F$ has $\varphi(\Delta_A) \subseteq V_\alpha$ for some α.

* Furthermore, if S is a manifold, we may further assume that $\varphi(\Delta_A)$ is a polygon so local Gauss-Bonnet applies to it.

Proof Lect 31-33

Global Gauss-Bonnet

Rem. Thm below also holds in non-oriented case.

Thm 3 Let S be a compact conn. oriented Riem. 2-fold. Then

$$\iint_S K_{Gauss}\, dS = 2\pi \chi(S).$$

Proof In notn thm 2, apply local G-B to each $\varphi(\Delta_A)$ to get

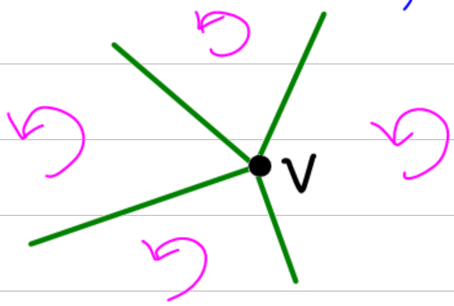
$$\iint_{\varphi(\Delta_A)} K_{Gauss}\, dS + \int_{\partial\varphi(\Delta_A)} k_g\, ds + \sum_j \alpha_{A,j} = 2\pi.$$

Summing over all A & using S oriented &

Lect. 23, Prop. get

$$\iint_S \kappa_{Gauss}\, dS + \sum \alpha_{A,j} \overset{(*)}{=} 2\pi |F|.$$

For each vertex v,



let n_v = no. edges with head v & $\alpha_{v,j}$ be ext. angles at v. Summing interior angles

$$\sum_j (\pi - \alpha_{v,j}) = 2\pi$$

$$\Rightarrow \sum_j \alpha_{v,j} = (n_v - 2)\pi.$$

Summing over all v gives

$$\sum_{v,j} \alpha_{v,j} = \sum_{A,j} \alpha_{A,j} = \sum_v (n_v - 2)\pi$$

$$= \sum n_v \pi - 2\pi |V|$$

$$= 2|E|\pi - 2\pi |V|$$

by graph theory.

Sub. into $(*)$ to conclude proof. $\square$

Eg 2 For the flat torus S: $\kappa_{Gauss} = 0$ so $\iint_S \kappa_{Gauss}\, dS = 0$. Also $\chi(S) = 0$ so

G-B verified directly here.

If $S \subset \mathbb{R}^3$ is torus obtained by rotating circle about an axis, G-B still gives $\iint_S K_{Gauss}\, dS = 0$, but

now K_{Gauss} is sometimes positive, sometimes negative!

$\boxed{\text{Classification of Compact Surfaces}}$

Thm 4 Any conn. compact surface is homeo to one of the following:

a Sphere. $\chi(S) = 2$, orientable

b Sphere with g handles $|K|$ where K has one face pair $A^{\pm 1}$ &

$$\partial A = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}.$$

$$\chi(S) = 2 - 2g, \text{ orientable.}$$

c Surface with k crosscaps $|K|$ where K has one face pair $A^{\pm 1}$ with

$$\partial A = a_1 a_1 a_2 a_2 \ldots a_k a_k.$$

$$\chi(S) = 2 - k, \text{ not orient.}$$

Proof Lect 29-30 show S homeo. to one on list. Orient. clear. For χ note, ex $|V| = 1$ so

(b) $\chi = |F| - |E| + |V| = 1 - 2g + 1 = 2 - 2g$

(c) $\chi = 1 - k + 1 = 2 - k$.

To check $|V|=1$, consider picture e.g. in case ⓑ