Lecture 27: Euler characteristic. Orientability

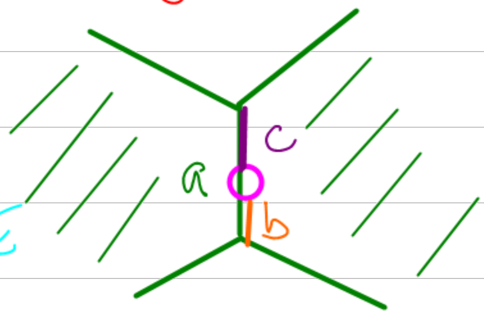
Aim Intro. combinatorial invariants which will be useful for studying top. of $|K|$ $\because$ they are top. invariants!

K = cell complex

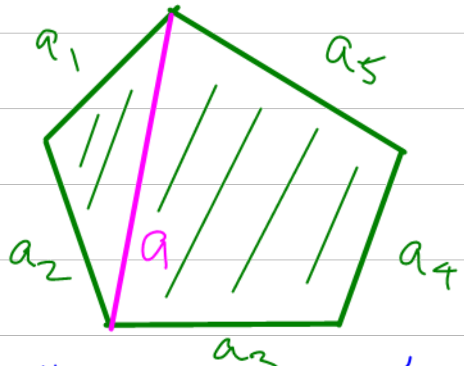
Elementary Subdivisions

DO NOT ERASE

Edge



Face



We can create new cell complex K' from K by one of following two elementary operations:

Defn 1 (E) Replace some edge $e \in E$ with two new edges b, c (& e^{-1} with c^{-1}, b^{-1}). Replace occurrence of e (resp. e^{-1}) in boundary of any face with bc (resp. $c^{-1}b^{-1}$).

Here we subdivide edge e.

(F) Subdivide face $A \in F$. Replace A with two new faces B, C (& A^{-1} with B^{-1}, C^{-1}). Add new edge a. Let $\partial A = a_1 \dots a_n$ & pick $i \in \{1, \dots, n\}$. Define $\partial B = a_1 \dots a_i a$, $\partial C = a^{-1} a_{i+1} \dots a_n$.

Defn 2 A refinement of K is a cell complex obtained from K by subdividing edges & faces &

possibly renaming faces/edges with different labels.

Two cell complexes K, K' are homeomorphic if there's a sequence of cellcomplexes $K = K_0, K_1, \dots, K_n = K'$ s.t. either K_{i+1} is a refinement of K_i or vice versa.

Clearly

Fact (a) K, K' homeomorphic $\Rightarrow |K|, |K'|$ homeo.

(b) The reln of being homeo. is an equiv. reln.

$\boxed{\text{Euler Characteristic}}$

Defn 3 The Euler characteristic of K is $\chi(K) := |F| - |E| + |V|$.

$|F|$ " no. face pairs, $|E|$ " no. edgepairs

N.B $|F|$ not no. faces & $|E|$ not no. edges!

Prop 1 K, K' homeo. $\Rightarrow \chi(K) = \chi(K')$.

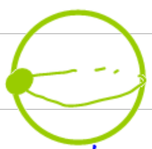
Proof By induction, suff. show subdividing edges (SE), or faces (SF) doesn't change χ.

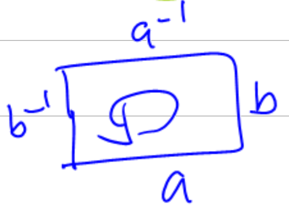
(SE) F unchanged but both $|E|$ & $|V|$ increased by 1.

(SF) V unchanged but both $|F|$ & $|E|$ increased by 1. $\square$

Rem Hence say χ invariant wrt. refinement

E.g.

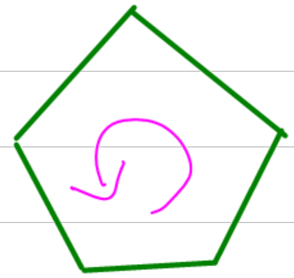




	$\lvert F\rvert$	$\lvert E\rvert$	$\lvert V\rvert$	χ
S^2	2	1	1!	2
T^2	1	2	1	0

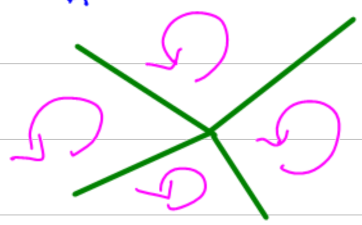
Orientability

Consider n-gon $\Delta_A = \Delta_{A^{-1}}$ assoc. to face $A \in F$.

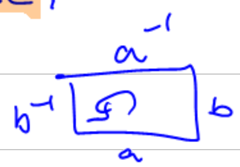


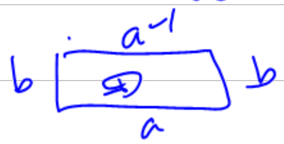
An orientation of Δ_A means embedding it in $\mathbb{R}^3$ & choosing a "side" OR equiv (by RH-rule) drawing $\circlearrowleft$ or $\circlearrowright$ on Δ_A $\Longleftrightarrow$ picking A or A^{-1}.

This agrees with notion of orientation for manifold Δ_A^{int} in Lect. 23.



Defn 4 An **orientation** of K is $O \subset F$ containing exactly one face from each face pair s.t. the boundaries of the $A \in O$ contain all the edges exactly once. If such O exists, we say K is **orientable**.

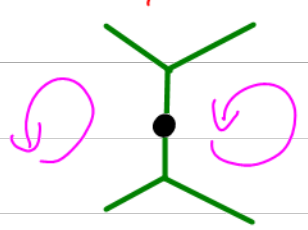
Eg $\mathbb{T}^2$ 

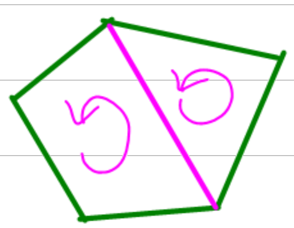
orientable but Klein bottle 

is not.

Prop 2 Suppose K, K' homeo. Then K is orientable iff K' is.

Proof by Picture





$\square$

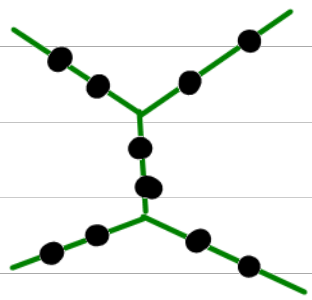
Rem An orientation of K gives a positive sense $\circlearrowleft$ about any $p \in |K|$ & this varies continuously with p.

Triangulation

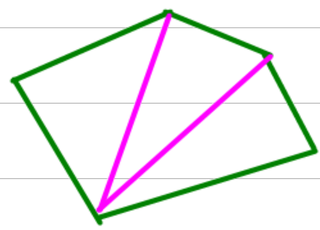
Defn 5 A triangulation of a top. surface S is an homeo. $\varphi : |K| \xrightarrow{\sim} S$ for some simplicial cell complex K. If such φ exists, we say S is triangulable.

Prop 3 Any cell complex K has a simplicial refinement. Hence $|K|$ is triangulable.

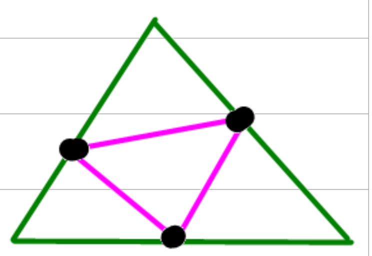
Proof First subdivide all edges twice so they're determined by their head & tail & no 1 or 2-gons.



Subdivide faces into 3-gons as shown.



Refine 3-gon faces as shown to ensure they're determined by boundary & new edges determined by head & tail.



$\square$