Lecture 25: Gluing Construction

Aim Construct top. spaces by gluing simpler ones together.

Gluing

Let X = top. space, $\sim$ an equiv. reln on X & $X/\sim = \{[x] \mid x \in X\}$ the set of equiv. classes. There's a quotient map $\pi : X \to X/\sim : x \mapsto [x]$ which is cont. $\because$ by defn $U \subseteq X/\sim$ is open iff $\pi^{-1} U$ is. Recall universal property

Prop 1 Let $f : X \to Y$ be cont. Suppose $x \sim x' \Rightarrow f(x) = f(x')$ so $\bar{f} : X/\sim \to Y : [x] \mapsto f(x)$ is a well-defined map of sets. Then $\bar{f}$ is cont. & $f = \bar{f} \circ \pi$.

Proof $U \subseteq Y$ open $\Rightarrow f^{-1} U = \pi^{-1} \bar{f}^{-1} U$ open $\Rightarrow$ $\bar{f}^{-1} U$ open. $\square$

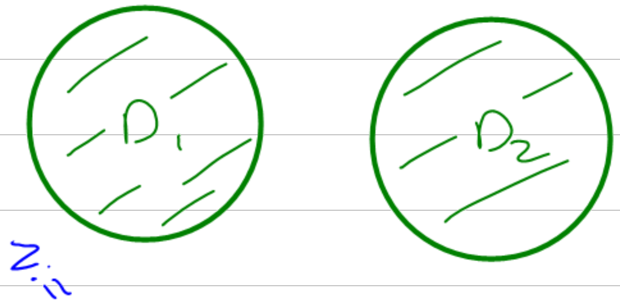
Setup Let $Z \subseteq X$ be closed & $\varphi : Z \to X$ be cont. map s.t.

* $\varphi^{-1}(z)$ is finite.

Define $\sim$ to be equiv. reln on X gen. by $z \sim \varphi(z)$ $\forall z \in Z$, i.e. $x \sim x'$ if $\exists$ chain of elts $x = x_1, x_2, \dots, x_k = x'$ s.t. $x_{i+1} = \varphi(x_i)$ or $x_i = \varphi(x_{i+1})$.

Defn Gluing X along φ is the formation of the quotient space $X/\varphi := X/\sim$

Eg 1 Let $X = D_1 \dot\cup D_2$ be disjoint union of two homeo. discs in $\mathbb{R}^2$.



Let $\varphi : \partial D_1 \to \partial D_2$ be natural homeo. Then $X/\varphi \simeq S^2$ since can map D_1, D_2 to upper & lower hemisphere. Prop 1 $\Rightarrow$ get cont. bij. $X/\varphi \to S^2$. Homeo $\because$ X/φ compact & S^2 Haus.

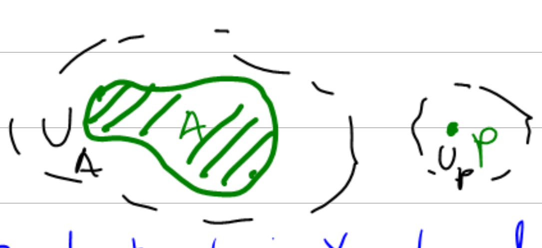
Prop 2 If X is compact Hausdorff, then $\pi : X \to X/\varphi$ is closed i.e. maps closed sets to closed sets.

Proof Given $C \subseteq X$ closed, suff show $\pi^{-1}\pi C$ closed. But $\pi^{-1}\pi C = \{c' \mid c' \sim c, \text{ for some } c \in C\}$ is finite union of following closed sets $C, \varphi^{-1}(C), \varphi(C \cap Z), \varphi^{-1}\varphi(C \cap Z)$ & some Y contained in finite set.

$\varphi^{-1}(z) \cup \varphi\varphi^{-1}(z) \cup \varphi^2\varphi^{-1}(z) \cup \varphi^{-1}\varphi^2\varphi^{-1}(z)$

ex. prove this! $\square$

Normality

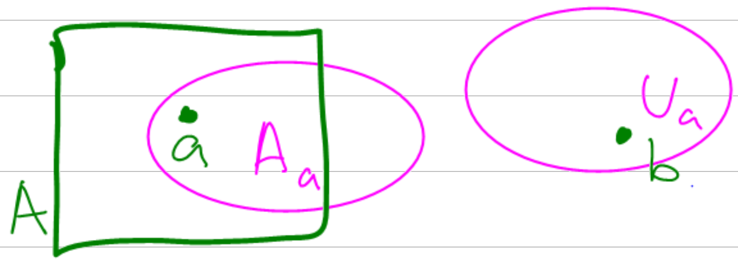


Defn 2 Suppose 1 pt sets in X closed e.g. X Hausdorff. We say X is *regular* (resp. *normal*) if for any closed $A \subseteq X$, $p \in X$ disjoint (resp. disjoint closed $A, B \subseteq X$) there are disjoint open

sets U_A, U_p (resp. U_A, U_B) s.t. $U_A \supseteq A$, $U_p \ni p$ (resp. $U_A \supseteq A$, $U_B \supseteq B$).

Prop 3 Any compact Hausdorff space X is normal.

Proof We first show X regular so ass. $A \subseteq X$ closed & $b \notin A$. X Hausdorff $\Rightarrow$ for each $a \in A$ $\exists$ disjoint open nbhds $A_a \ni a$, $U_a \ni p$



A also compact so can cover A with $U_A = A_{a_1} \cup \dots \cup A_{a_m}$. Then $U_b = \bigcap U_{a_i}$ is open nbhd of b & $U_b \cap U_A = \emptyset$.

To show normal, repeat above argument varying now b over a closed $B \subseteq X$ disjoint from A & using regularity to obtain disjoint open nbhds of A & b.

$\square$

$\boxed{\text{Properties of Gluing Construction}}$

Lemma Let X be a normal space & $\pi : X \to Y$ a quotient map i.e. π is surj. &

Y has quotient top. If π is closed, then Y is normal.

Proof: π surj. $\Rightarrow$ any 1 pt set $\{y\} \subseteq Y$ is the image of a 1 pt set in X, Hence π closed $\Rightarrow \{y\}$ closed.

Let $A, B \subseteq Y$ be disjoint closed so $\pi^{-1}A, \pi^{-1}B \subseteq X$ are disjoint closed. X normal $\Rightarrow \exists$ disjoint open $U_A \supseteq \pi^{-1}A$, $U_B \supseteq \pi^{-1}B$.

We claim $V_A := \pi(U_A{}^c)^c$, $V_B := \pi(U_B{}^c)^c$ are disjoint opens containing A, B resp.

Now π closed $\Rightarrow V_A, V_B$ open.

$$A \subseteq V_A \because V_A{}^c = \pi(U_A^c) \subseteq \pi((\pi^{-1}A)^c) \overset{ex}{\subseteq} A^c.$$

$$V_A \cap V_B = (\pi(U_A{}^c) \cup \pi(U_B{}^c))^c$$

$$\subseteq (\pi(\underbrace{U_A^c \cup U_B^c}_{= X}))^c = Y^c = \emptyset.$$

$\square$

Cor Let X/φ be top. space obtained from gluing a compact Hausdorff space X along some cont. $\varphi: Z \to X$. Then X/φ is compact Hausdorff.

Proof Prop. 2, 3 & lemma $\Rightarrow X/\varphi$ normal & $\therefore$ Hausdorff.

Also $\pi: X \to X/\varphi$ surj. so X compact $\Rightarrow X/\varphi$ compact. $\square$