# Lecture 25: Gluing Construction

**Aim** Construct top. spaces by gluing simpler ones together.

## Gluing

Let $X$ = top. space, $\sim$ an equiv. reln on $X$ & $X/\sim = \{[x] \mid x \in X\}$ the set of equiv. classes. There's a quotient map $\pi : X \to X/\sim : x \mapsto [x]$ which is cont. $\because$ by defn $U \subseteq X/\sim$ is open iff $\pi^{-1}U$ is. Recall universal property

**Prop 1** Let $f : X \to Y$ be cont. Suppose $x \sim x' \Rightarrow f(x) = f(x')$ so $\bar{f} : X/\sim \to Y : [x] \mapsto f(x)$ is a well-defined map of sets. Then $\bar{f}$ is cont. & $f = \bar{f} \circ \pi$.

**Proof** $U \subseteq Y$ open $\Rightarrow f^{-1}U = \pi^{-1}\bar{f}^{-1}U$ open $\Rightarrow \bar{f}^{-1}U$ open. $\square$

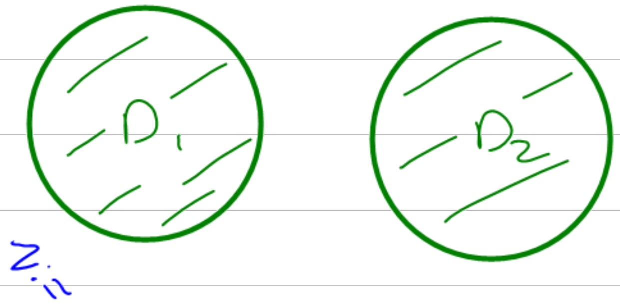
**Setup** Let $Z \subseteq X$ be closed & $\varphi : Z \to X$ be cont. map s.t.

* $\varphi^{-1}(z)$ is finite & $\varphi^{-1}(x)$ finite for all $x \in X$.

Define $\sim$ to be equiv. reln on $X$ gen. by $z \sim \varphi(z)$ $\forall z \in Z$, i.e. $x \sim x'$ if $\exists$ chain of elts $x = x_1, x_2, \ldots, x_k = x'$ s.t. $x_{i+1} = \varphi(x_i)$ or $x_i = \varphi(x_{i+1})$.

**Defn** Gluing $X$ along $\varphi$ is the formation of the quotient space $X/\varphi := X/\sim$

**Eg 1** Let $X = D_1 \dot\cup D_2$ be disjoint union of two homeo. discs in $\mathbb{R}^2$.




Let $\varphi : \partial D_1 \to \partial D_2$ be natural homeo. Then $X/\varphi \simeq S^2$ since can map $D_1, D_2$ to upper & lower hemisphere. Prop 1 $\Rightarrow$ get cont. bij. $X/\varphi \to S^2$. Homeo $\because$ $X/\varphi$ compact & $S^2$ Haus.

**Prop 2** If $X$ is compact Hausdorff, then $\pi : X \to X/\varphi$ is closed i.e. maps closed sets to closed sets.

**Proof** Given $C \subseteq X$ closed, suff show $\pi^{-1}\pi C$ closed. But $\pi^{-1}\pi C = \{c' \mid c' \sim c, \text{ for some } c \in C\}$ is finite union of following closed sets $C, \varphi^{-1}(C), \varphi(C \cap Z), \varphi^{-1}\varphi(C \cap Z)$ & some $Y$ contained in finite set.

$\varphi^{-1}(z) \cup \varphi\varphi^{-1}(z) \cup \varphi^2\varphi^{-1}(z) \cup \varphi^{-1}\varphi^2\varphi^{-1}(z)$

ex. prove this! $\square$

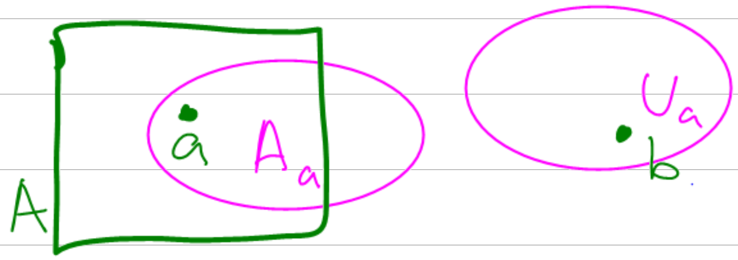
## Normality


**Defn 2** Suppose 1 pt sets in $X$ closed e.g. $X$ Hausdorff. We say $X$ is regular (resp. normal) if for any closed $A \subseteq X$, $p \in X$ disjoint (resp. disjoint closed $A, B \subseteq X$) there are disjoint open

sets $U_A, U_p$ (resp. $U_A, U_B$) s.t. $U_A \supseteq A, U_p \ni p$ (resp. $U_A \supseteq A, U_B \supseteq B$).

**Prop 3** Any compact Hausdorff space $X$ is normal.

**Proof** We first show $X$ regular so ass. $A \subseteq X$ closed & $b \notin A$. $X$ Hausdorff $\Rightarrow$ for each $a \in A$ $\exists$ disjoint open nbhds $A_a \ni a$, $U_a \ni p$




$A$ also compact so can cover $A$ with $U_A = A_{a_1} \cup \dots \cup A_{a_m}$. Then $U_b = \bigcap U_{a_i}$ is open nbhd of $b$ & $U_b \cap U_A = \emptyset$.

To show normal, repeat above argument varying now $b$ over a closed $B \subseteq X$ disjoint from $A$ & using regularity to obtain disjoint open nbhds of $A$ & $b$.

$\square$

## Properties of Gluing Construction

**Lemma** Let $X$ be a normal space & $\pi: X \to Y$ a quotient map i.e. $\pi$ is surj. &

$Y$ has quotient top. If $\pi$ is closed, then $Y$ is normal.

Proof: $\pi$ surj. $\Rightarrow$ any 1 pt set $\{y\} \subseteq Y$ is the image of a 1 pt set in $X$, Hence $\pi$ closed $\Rightarrow \{y\}$ closed.

Let $A, B \subseteq Y$ be disjoint closed so $\pi^{-1}A, \pi^{-1}B \subseteq X$ are disjoint closed. $X$ normal $\Rightarrow \exists$ disjoint open $U_A \supseteq \pi^{-1}A$, $U_B \supseteq \pi^{-1}B$.

We claim $V_A := \pi(U_A{}^c)^c$, $V_B := \pi(U_B{}^c)^c$ are disjoint opens containing $A, B$ resp.

Now $\pi$ closed $\Rightarrow V_A, V_B$ open.

$$A \subseteq V_A \because V_A{}^c = \pi(U_A^c) \subseteq \pi\left((\pi^{-1}A)^c\right) \overset{ex}{\subseteq} A^c.$$

$$V_A \cap V_B = \left(\pi(U_A{}^c) \cup \pi(U_B{}^c)\right)^c$$

$$\subseteq \left(\pi(\underbrace{U_A^c \cup U_B{}^c}_{= X})\right)^c = Y^c = \emptyset.$$

$\square$

Cor Let $X/\varphi$ be top. space obtained from gluing a compact Hausdorff space $X$ along some cont. $\varphi: Z \to X$. Then $X/\varphi$ is compact Hausdorff.

Proof Prop. 2, 3 & lemma $\Rightarrow X/\varphi$ normal & $\therefore$ Hausdorff.

Also $\pi: X \to X/\varphi$ surj. so $X$ compact $\Rightarrow X/\varphi$ compact. $\square$