Lecture 24: Integration on 2D-Riemannian Manifold

Aim Define integration on abstract 2D-Riem. manifold (M, G) so local Gauss-Bonnet holds in this general setting.

Material here sketchier. Aim to see how Lectures 15 & 16 gen. to abstract surface setting.

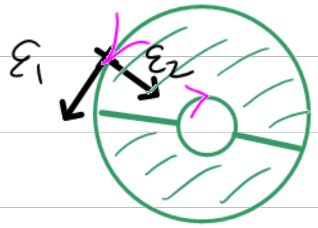
$\boxed{\text{Polygons in Abstract 2D-manifolds}}$

c.f. Defn 1, Lect. 15.

Defn 1 a A polygon P on M is a closed subset $P \subseteq M$ which is homeomorphic to the closed unit disk in $\mathbb{R}^2$ & ∂P is a p-wise smooth curve (wrt some param.), with exterior angles $< \pi$.

b $D \subseteq M$ is a domain if it's a finite union of polygons P_i, $i \in I$ s.t. $P_i \cap P_j \subseteq \partial P_i \cap \partial P_j$ whenever $i \neq j$.

Eg. The annulus in $\mathbb{R}^2$ is a domain being union of two polygons as in picture.



orientation

Fact If D is a domain in an oriented 2D-manifold M, then ∂D, say param. by p-wise smooth curves $\gamma_j : I_j \to M$, is naturally oriented as follows:

* If $\varepsilon_1, \varepsilon_2$ are Frenet frame for γ_j, then ε_2 points to interior of D.

i.e. if surface orientation $\circlearrowleft$, then orientation ∂D also $\circlearrowleft$.

$\boxed{\text{Integration}}$

Disclaimer We won't set up integration theory properly (via partitions of unity). We instead assume naively that domains $D \subseteq M$ can be chopped up nicely into subdomains D_i contained in charts where we can integrate as follows. This fine in practice.

Prop-Defn Let $\varphi : V \to \mathbb{R}^2_u$, $\tilde{\varphi} : \tilde{V} \to \mathbb{R}^2_{\tilde{u}}$ be two charts & $D \subset V \cap \tilde{V}$ a domain. Let $\underline{G}$, $\underline{\tilde{G}}$ be matrices assoc. to G wrt φ & $\tilde{\varphi}$. Then for any cont. $f : M \to \mathbb{R}$ we have

$$\int_{\varphi(D)} f(\varphi^{-1}(u)) \sqrt{\det \underline{G}(u)} \, du^1 du^2$$

$$= \int_{\tilde{\varphi}(D)} f(\tilde{\varphi}^{-1}(\tilde{u})) \sqrt{\det \underline{\tilde{G}}(\tilde{u})} \, d\tilde{u}^1 d\tilde{u}^2$$

We denote the common value $\int_D f \, dM$.

Rem. This gen. MATH2111 calculation of surface integral via double integral: given embedded param. surface $\sigma : U \xrightarrow{\sim} S \subset \mathbb{R}^3$,

ex $\sqrt{\det \underline{G}} = |\partial_1 \sigma \times \partial_2 \sigma|$.

Proof Let $\tilde{u} = \psi(u)$ be the transition fn.

Formula 1' Lect. 8 $\Rightarrow$

$$\underline{G} = (d\psi)^T \underline{\tilde{G}} (d\psi)$$

Calculus change of var. $\Rightarrow$

$$d\tilde{u}^1 d\tilde{u}^2 = |\det(d\psi)| du^1 du^2$$

$\therefore$

$$\sqrt{\det(\underline{G})}\, du^1 du^2 = \sqrt{\det(d\psi)^2 \det(\underline{\tilde{G}})}\, du^1 du^2$$

$$= \sqrt{\det(\underline{\tilde{G}})}\, d\tilde{u}^1 d\tilde{u}^2 \quad = d\tilde{M}$$

$\square$

Local Gauss-Bonnet Gen. manifold form.

Let M be an orient. 2-dim Riem. man. & $P \subseteq M$ be a polygon with a finite no. of corners $\{p_1, \ldots, p_n\}$ where ∂P not smooth.

Fact $\Rightarrow$ ∂P is an oriented p-wise smooth curve. We can unit speed par. it via arclengths in pos. dirn. Orient. $\Rightarrow$ have oriented exterior angles $\alpha_j \in (-\pi, \pi)$ at p_j.

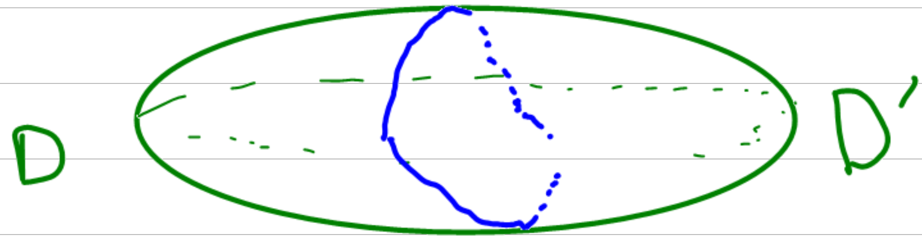
Thm $$\iint_P K_{Gauss}\, dM + \int_{\partial P} \kappa_g\, ds + \sum \alpha_j = 2\pi.$$

No Proof Proved for subman. of $\mathbb{R}^3$ in Lect. 16. Same proof in general, but deform to flat case by deforming G. $\square$

Prelude to Global Gauss-Bonnet Theorem

Consider ellipsoid $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \subset \mathbb{R}^3$.

$\partial D = C$, $\partial D' = -C$



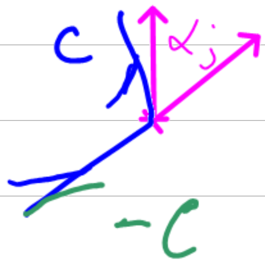
Cut S as shown into two polygons D, D' so ∂D = orient. curve C say & $\partial D' = -C$:= curve C with reverse orient.

Prop a $\kappa_g^{-C}(p) = -\kappa_g^{C}(p)$.

b If α_j (resp. α_j') is ext. angle of C (resp. $-C$) at p_j, then $\alpha_j' = -\alpha_j$.

Proof a If C param. by t, can ass. $-C$ param. by $-t$. If $\varepsilon_1, \varepsilon_2$ Frenet frame for C, then $-\varepsilon_1, -\varepsilon_2$ Frenet frame for $-C$.

$$\kappa_g^{-C} = G(\nabla_{-t}(-\varepsilon_1), -\varepsilon_2) = -G(\nabla_t \varepsilon_1, \varepsilon_2) = -\kappa_g^{C}.$$

b 

See picture.

$\square$

Local G-B applied to D & D' + Prop. $\Rightarrow$

Thm For any ellipsoid S, $\iint_S \kappa_{Gauss}\, dS = 4\pi$.

Rem We see from the argument that this thm extends to much more gen. surfaces, in fact anything homeo. to a sphere!