Lecture 22: 2D – Riemannian Manifolds

Aim: Generalise notions of geodesics & curvature to 2-dim Riem. man (M, G).

Material in next 3 lectures covered in less depth. Key point: most results of lectures 7-16 hold for arbitrary 2D-Riem. man. with appropriate defns.

Abstract Definitions

Let $\varphi: V \longrightarrow \mathbb{R}^2_u$ be a chart on M & (G_{ij}) coord. of G wrt. φ. Lect. 12, Thm 1 suggests

Defn 1 The Christoffel symbols (wrt φ) are defined to be

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l G^{kl} \left(\partial_i G_{lj} + \partial_j G_{li} - \partial_l G_{ij} \right)$$

where (G^{kl}) is the inverse $(2\ 0)$-tensor to (G_{ij}).
& $\partial_l G_{ij} = \frac{\partial}{\partial u^l}\left(G_{ij}\left(\varphi^{-1}(u^1, u^2)\right)\right)$.

Eg 1 Hyp. plane H^2_r. Coords u, v.

$$(G_{ij}) = \begin{pmatrix} r^2 v^{-2} & 0 \\ 0 & r^2 v^{-2} \end{pmatrix} = \frac{r^2}{v^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (G^{kl}) = \frac{v^2}{r^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \partial_2 G_{ii} = -\frac{2r^2}{v^3}$$

$$\therefore \Gamma^k_{ij} = \frac{v^2}{2r^2}\left(\partial_i G_{kj} + \partial_j G_{ki} - \partial_k G_{ij}\right)$$

$$= 0 \quad \text{if} \quad \{i, j, k\} = \{1, 1, 1\} \text{ or } \{1, 2, 2\}$$

$$\Gamma^2_{22} = -\frac{1}{v}, \quad \Gamma^2_{11} = \frac{1}{v}, \quad \Gamma^1_{12} = -\frac{1}{v}.$$

Below consider par. curve $\gamma: I \longrightarrow V \subseteq M$ on M. Working in coords let $c(t) = \varphi(\gamma(t)) = (u^i(t))$.

Thm-Defn The following quantities/notions are independent of the choice of chart φ.

Defined in Lect. 19 Eg 2

(a) Consider vect. field $X(t) = \sum X^k(t)\frac{\partial}{\partial u^k}$ along γ in M.

The covariant derivative $\nabla_t X \in T_{\gamma(t)}M$ at $t \in I$ defined using co-ords by

$$\nabla_t X = \sum \dot{X}^k \frac{\partial}{\partial u^k} + \sum_{k,i,j} \Gamma^k_{ij} X^i \dot{u}^j \frac{\partial}{\partial u^k}$$

c.f. Formula 1 Lect.

(b) γ is a geodesic if $\nabla_t \dot{\gamma} = 0 \iff$

$0 = \ddot{u}^k + \sum \dot{u}^i \dot{u}^j \Gamma^k_{ij}$, $k = 1, 2$. c.f. Lect. 14, Defn 3

(c) The Gaussian curvature K_{Gauss} is the scalar fn on M computed via $\det(H)/\det(G)$, $\det(H)$ defined via Gauss's eqn (Lect 12 Thm 2) intrinsically in terms of (G_{ij}). If $(G_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & G_{22} \end{pmatrix}$, then $K_{Gauss} = \partial_1^2 \sqrt{G_{22}} / \sqrt{G_{22}}$ as in Lect. 12, Prop.

K_{Gauss} is a bending invariant.

Proof Omitted. Not exam. Long tedious calculation. □

$\boxed{\text{Hyperbolic Plane } H^2_r}$

$u^1 = u,\ u^2 = v$

E.g. 1 again. $\Gamma^2_{11} = v^{-1}$, $\Gamma^2_{22} = \Gamma^1_{12} = -v^{-1}$.

ODE for geodesic

$$0 \overset{*_u}{=} \ddot{u}^1 + \sum_{i,j} \Gamma^1_{ij} \dot{u}^i \dot{u}^j = \ddot{u} - \frac{2}{v}\dot{u}\dot{v}$$

squares here

$$0 \overset{*_v}{=} \ddot{u}^2 + \sum_{i,j} \Gamma^2_{ij} \dot{u}^i \dot{u}^j = \ddot{v} + \frac{1}{v}\dot{u}^2 - \frac{1}{v}\dot{v}^2$$

$\ddot{u} = 0 \Rightarrow *_u$. Then $*_v$ becomes

$0 = v^{-1}\ddot{v} - v^{-2}\dot{v}^2 = \frac{d}{dt}(v^{-1}\dot{v}) \Longleftrightarrow \dot{v} = \text{Const.}\, v$

A unit speed geodesic thru $(u_0, 1)$ at $t=0$ is $v(t) = e^{-t/r}$, $u = u_0$ const. $\because$ square speed is

$$G\left(\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}, \begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}\right) = \dot{v}(t)\, G_{22}\, \dot{v}(t) = \frac{1}{r^2} e^{-2t/r} \cdot \frac{r^2}{v^2} \underset{\text{at } t=0}{=\!=} 1$$

Prop 1 The geodesics on H^2_r are the vertical lines & semicircles centred on $v=0$ line.

Proof Isom. $(u,v) \mapsto \left(-\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$ maps

geodesic (a, v), $v > 0$ $\mapsto \left(x = -\frac{a}{a^2+v^2}, y = \frac{v}{a^2+v^2}\right)$

The latter lies in locus

$$x^2 + y^2 = \frac{a^2 + v^2}{(a^2+v^2)^2} = -a^{-1}x$$

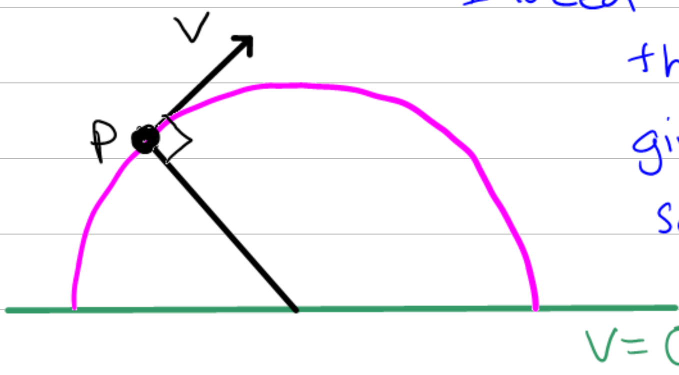
i.e. $\left(x + \frac{1}{2a}\right)^2 + y^2 = \frac{1}{4a^2}$

circle centre $\left(-\frac{1}{2a}, 0\right)$

Using horizontal translation isometry & scaling isom, Prop. 4 Lect. 21, we can map this semicircle to any semicircle with centre on $v=0$ line.

Isom. preserve geodesics, so these semicircles are all geodesics.

In fact, these semicircles & vertical lines are the only geodesics.



Indeed the unique geodesic thru p in drn v is given by building semicircle as in picture. $\square$

Prop 3 On H^2_r, $K_{Gauss} = -\frac{1}{r^2}$.

Rem Symmetry $\Rightarrow$ it's constant.

Proof Use geod. coords $(\tilde{u}, \tilde{v})$ based on curve $v=1$ ie. $\begin{pmatrix} u \\ v \end{pmatrix} = \varphi\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} := \begin{pmatrix} \tilde{v} \\ e^{-\tilde{u}/r} \end{pmatrix}$ so

$$d\varphi = \begin{pmatrix} 0 & 1 \\ -\frac{1}{r}e^{-\tilde{u}/r} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -v/r & 0 \end{pmatrix}$$

Lecture 8 Formula 1' $\Rightarrow$

$$\underline{\tilde{G}} = (d\varphi)^T \underline{G}\,(d\varphi) = \begin{pmatrix} 0 & -v/r \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{r^2}{v^2} & 0 \\ 0 & \frac{r^2}{v^2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\frac{v}{r} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & r^2/v^2 \end{pmatrix}$$

$$\therefore \sqrt{\tilde{G}_{22}} = \frac{r}{v} = re^{\tilde{u}/r} \qquad K_{Gauss} = -\frac{\partial_1^2\sqrt{\tilde{G}_{22}}}{\sqrt{\tilde{G}_{22}}} = -\frac{1}{r^2}$$

$\square$

$\boxed{\text{Hyperbolic Geometry}}$

- Angle sum of geod. Δ is $< \pi$ in agreement with local G-B.
- Playfair's axiom fails. We have non-Euclidean geometry.