

Lecture 22: 2D - Riemannian Manifolds

Aim Generalise notions of geodesics & curvature to 2-dim Riem. man (M, G) .

Material in next 3 lectures covered in less depth. Key point: most results of lectures 7-16 hold for arbitrary 2D-Riem. man. with appropriate detns.

Abstract Definitions

Let $\varphi: V \rightarrow \mathbb{R}^2_u$ be a chart on M & (G_{ij}) coord. of G wrt. φ . Lect. 12, Thm 1 suggests

Defn 1 The Christoffel symbols (wrt φ) are defined to be

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell} G^{\ell k} (\partial_i G_{\ell j} + \partial_j G_{\ell i} - \partial_{\ell} G_{ij})$$

where (G^{kl}) is the inverse (2,0)-tensor to (G_{ij}) & $\partial_{\ell} G_{ij} = \frac{\partial}{\partial u^{\ell}} (G_{ij}(\varphi^{-1}(u^1, u^2)))$.

Eg 1 Hyp. plane H^2_r . Coords u, v .

$$(G_{ij}) = \begin{pmatrix} r^2 v^2 & 0 \\ 0 & r^2 v^{-2} \end{pmatrix} = \frac{r^2}{v^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (G^{kl}) = \frac{v^2}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \partial_2 G_{ij} = -\frac{2r^2}{v^3}$$
$$\therefore \Gamma^k_{ij} = \frac{v^2}{2r^2} (\partial_i G_{kj} + \partial_j G_{ki} - \partial_k G_{ij})$$
$$\Gamma^2_{22} = -\frac{1}{v}, \Gamma^2_{11} = \frac{1}{v}, \Gamma^1_{12} = -\frac{1}{v}.$$

Below consider par. curve $\gamma: I \rightarrow V \subseteq M$ on M . Working in coords let $c(t) = \varphi(\gamma(t)) = (u^i(t))$.

Thm-Defn The following quantities/notions are independent of the choice of chart φ .

Defined in Lect. 19 Eg 2. \rightarrow

(a) Consider vect. field $X(t) = \sum X^k(t) \frac{\partial}{\partial u^k}$ along γ in M .

The covariant derivative $\nabla_t X \in T_{\gamma(t)}M$ at $t \in I$ defined using co-rds by

$$\nabla_t X = \sum \dot{X}^k \frac{\partial}{\partial u^k} + \sum_{k,ij} \Gamma_{ij}^k X^i \dot{u}^j \frac{\partial}{\partial u^k}$$

c.f. Formula 1 Lect.

(b) γ is a geodesic if $\nabla_t \dot{\gamma} = 0 \iff 0 = \ddot{u}^k + \sum u^i \dot{u}^j \Gamma_{ij}^k, k=1,2$. c.f. Lect. 14, Defn 3

(c) The Gaussian curvature K_{Gauss} is the scalar fn on M computed via $\det(H)/\det(G)$, $\det(H)$ defined via Gauss's eqn (Lect 12 Thm 2) intrinsically in terms of (G_{ij}) : If

$$(G_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & u_{22} \end{pmatrix}, \text{ then } K_{\text{Gauss}} = -\frac{\partial^2 \sqrt{u_{22}}}{\sqrt{u_{22}}} \text{ as in Lect. 12, Prop.}$$

K_{Gauss} is a bending invariant.

Proof Omitted. Not exam. Long tedious calculation. \square

Hyperbolic Plane $H^2_{\mathbb{R}}$

E.g. 1 again. $\Gamma_{11}^2 = v^{-1}, \Gamma_{22}^2 = \Gamma_{12}^1 = -v^{-1}$.
ODE for geodesic

$$0 \stackrel{*u}{=} \ddot{u}^1 + \sum_{i,j} \Gamma_{ij}^1 \dot{u}^i \dot{u}^j = \ddot{u} - \frac{2}{v} \dot{u} \dot{v}$$

squares here

$$0 \stackrel{*v}{=} \ddot{u}^2 + \sum_{i,j} \Gamma_{ij}^2 \dot{u}^i \dot{u}^j = \ddot{v} + \frac{1}{v} \dot{u}^2 - \frac{1}{v} \dot{v}^2$$

$$\ddot{u} = 0 \Rightarrow \dot{u} = \text{const. } u. \text{ Then } \dot{v} \text{ becomes}$$

$$0 = v^{-1} \ddot{v} - v^{-2} \dot{v}^2 = \frac{d}{dt} (v^{-1} \dot{v}) \Leftrightarrow \dot{v} = \text{const. } v$$

A unit speed geodesic thru $(u_0, 1)$ at $t=0$ is $v(t) = e^{-t/r}$, $u = u_0$ const. \therefore square speed is

$$G\left(\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}, \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}\right) = \dot{v}(t) G_{22} \dot{v}(t) = \frac{1}{r^2} e^{-2t/r} \cdot \frac{r^2}{v^2} \Big|_{t=0} = 1$$

Prop 1 The geodesics on H_r^2 are the vertical lines & semicircles centred on $v=0$ line.

Proof Isom. $(u, v) \mapsto \left(-\frac{u}{a^2+v^2}, \frac{v}{a^2+v^2}\right)$ maps
 geodesic (a, v) $\mapsto \left(x = -\frac{a}{a^2+v^2}, y = \frac{v}{a^2+v^2}\right)$
 $v > 0$

The latter lies in locus

$$x^2 + y^2 = \frac{a^2 + v^2}{(a^2 + v^2)^2} = -a^{-1}x$$

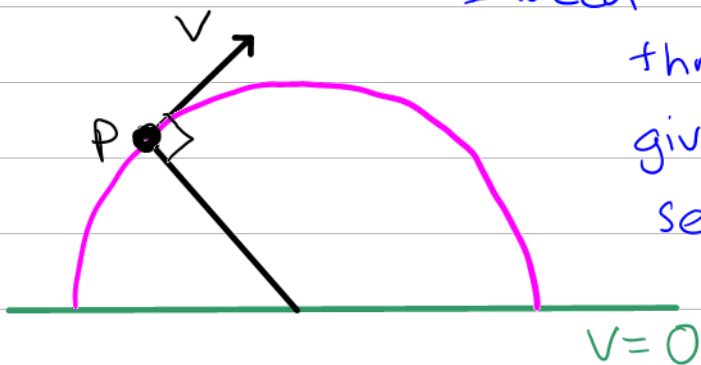
$$\text{i.e. } \left(x + \frac{1}{2a}\right)^2 + y^2 = \frac{1}{4a^2}$$

circle centre $\left(-\frac{1}{2a}, 0\right)$

Using horizontal translation isometry & scaling isom, Prop. 4 Lect. 21, we can map this semicircle to any semicircle with centre on $v=0$ line.

Isom. preserve geodesics, so these semicircles are all geodesics.

In fact, these semicircles & vertical lines are the only geodesics.



Indeed the unique geodesic thru p in dir'n v is given by building semicircle as in picture.

□

Prop 3 On H_r^2 , $K_{\text{Gauss}} = -\frac{1}{r^2}$.

Rem Symmetry \Rightarrow it's constant.

Proof Use geod. coords (\tilde{u}, \tilde{v}) based on curve $v=1$ i.e. $\begin{pmatrix} u \\ v \end{pmatrix} = \varphi \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} := \begin{pmatrix} \tilde{v} \\ e^{-\tilde{u}/r} \end{pmatrix}$ so

$$d\varphi = \begin{pmatrix} 0 & 1 \\ -\frac{1}{r}e^{-\tilde{u}/r} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{r} & 0 \end{pmatrix}$$

Lecture 8 Formula 1' \Rightarrow

$$\begin{aligned} \tilde{\underline{G}} &= (d\varphi)^T \underline{G} (d\varphi) = \begin{pmatrix} 0 & -1/r \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1/r & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & r^2/r^2 \end{pmatrix} \end{aligned}$$

$$\therefore \sqrt{\tilde{G}_{22}} = \frac{r}{r} = 1 \quad K_{\text{Gauss}} = -\frac{\partial_1^2 \sqrt{\tilde{G}_{22}}}{\sqrt{\tilde{G}_{22}}} = -\frac{1}{r^2}$$

□

Hyperbolic Geometry

* Angle sum of geod. Δ is $< \pi$ in agreement with local G-B.

* Playfair's axiom fails. We have non-Euclidean geometry.