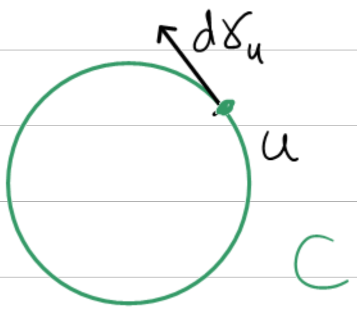
# Lecture 2: Curves

**Aim Lecture** Look at basic notions for 1-folds.

Below intervals can be closed, open or half-open.

**Defn 1** A (smooth) parametrised curve is a parametrised 1-fold or more gen. map $\gamma : I \to C \subseteq \mathbb{R}^m$ where $I$ is an interval & $\gamma$ extends to param. 1-fold $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^m$ for some open interval $\tilde{I} \supseteq I$.

**E.g.** $\gamma : [0, 2\pi] \to \mathbb{R}^2 : u \mapsto (\cos u, \sin u)^T$



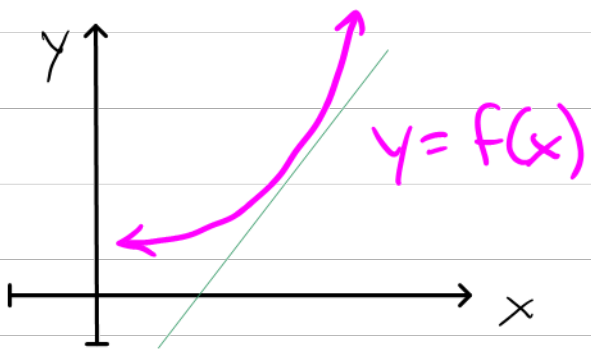

$d\gamma_u = \begin{pmatrix} -\sin u \\ \cos u \end{pmatrix} \neq 0$ for any $u$ so injective $\forall u$.

Technically this is linear map $\mathbb{R} \to \mathbb{R}^2$ represented by the $2\times 1$-matrix above. N.B. $n \times 1$-matrix is injective iff it's non-zero.

$\boxed{\text{Inverse Function Thm}}$ Recall

**Thm** Let $U \overset{open}{\subseteq} \mathbb{R}^n$ & $f : U \to \mathbb{R}^m$ be smooth. Then $f$ is a local diffeo. at $u \in U$ iff $df_u$ is an isomorphism (i.e. bijective linear map).

**e.g.**



# Reparametrisation

**Defn 2** Let $I \subseteq \mathbb{R}$ be a not nec. open interval. A fn $f: I \to \mathbb{R}^m$ is smooth if it extends to a smooth fn $\tilde{f}: \tilde{I} \to \mathbb{R}^m$ on an open interval $\tilde{I} \supseteq I$.

Sim, a surj. map of intervals $f: I \to I'$ is a diffeomorphism if it extends to a diffeo on open nbhds of $I, I'$.

**Cor** Consider smooth $f: I \to \mathbb{R}$ with $I$ an interval. Then $f: I \to f(I)$ is a diffeo. iff $df_p \neq 0 \ \forall p \in I$. In this case, either $df > 0$ or $df < 0$. means $df_p > 0 \ \forall p$.

**Proof** ($\Leftarrow$) Suppose $df_p \neq 0 \ \forall p$. Now $I$ conn. $\Rightarrow$ either $df > 0$ or $df < 0$ so $f$ is either increasing or decr. & thus induces bij$^n$ $f: I \to f(I)$. Inverse fn thm $\Rightarrow$ $f^{-1}$ is smooth so $f$ is a diffeo. $\square$

**Prop-Defn 1:** Let $\gamma: I \twoheadrightarrow C$ be a param. curve. A change of variables is a diffeo $\phi: \tilde{I} \to I$ where $\tilde{I}$ is an interval.

In this case, the reparametrisation wrt. $\phi$: $\tilde{\gamma} := \gamma \circ \phi : \tilde{I} \xrightarrow{\sim} I \to C$ is a param. curve.

We say $\phi$ is orientation preserving

if $d\phi > 0$ & orientation reversing if $d\phi < 0$.

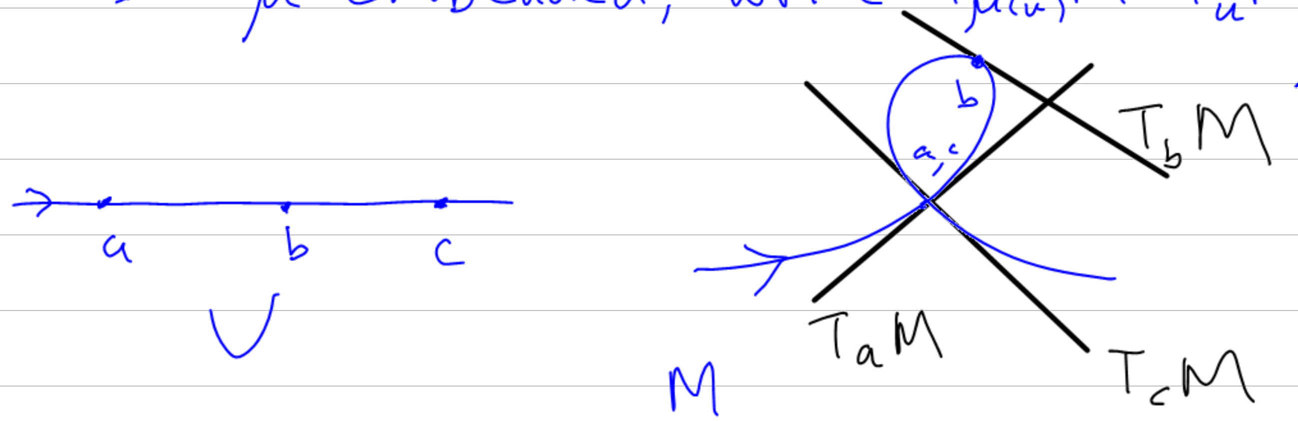
Rem: latter trace trajectory in reverse dir$^{\underline{n}}$.

Proof For $\tilde{p} \in \tilde{I}$, chain rule $\Rightarrow d\tilde{\gamma}_p = d\gamma_{\phi(\tilde{p})} \circ d\phi_{\tilde{p}}$ which is inj. $\because d\gamma$ & $d\phi$ are. $\square$

## Tangent Spaces to n-folds

Given a par. n-fold $\mu : U \twoheadrightarrow M \subseteq \mathbb{R}^m$ and $u \in U$, the tangent space at $u$ is the subspace $T_u M := d\mu_u(T_u\mathbb{R}^n) \leq T_{\mu(u)}\mathbb{R}^m$.

If $\mu$ embedded, write $T_{\mu(u)}M = T_u M$.




For par. curve $\gamma : I \twoheadrightarrow C$, $\dot{\gamma}(u) := d\gamma_u(e_1) \in T_u C$.

means defined to be

$e_1 \in T_u\mathbb{R} = \mathbb{R}$ so $e_1 = 1$.

E.g. $\gamma : \mathbb{R} \twoheadrightarrow C \subset \mathbb{R}^2$ defined by $\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, $\dot{\gamma}(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$T_{(1,1)}C = \mathbb{R}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \subset T_{(1,1)}\mathbb{R}^2 \cong \mathbb{R}^2$

# Arc Length

Doesn't respect Einstein's rule but $\sum a^i b^j \delta_{ij}$ does.

**Notn** for usual dot product

$$\langle \sum a^i e_i, \sum b^i e_i \rangle = \sum a^i b^i$$

$$|v|^2 = \langle v, v \rangle \quad , v \in \mathbb{R}^n .$$

**Recall** the arc length of a par. curve $\gamma : I \twoheadrightarrow C$ is $\int_I |\dot{\gamma}(u)| \, du$.

**Defn 3** We say $\gamma : I \twoheadrightarrow C$ is parametrised by arc length or is unit speed if $|\dot{\gamma}(u)| = 1$ for all $u \in I$.

**Prop** Any par. curve $\gamma : I \twoheadrightarrow C$ has a unit speed reparametrisation.

**Proof** Pick $a \in I$ and define new parameter

$$s = \int_a^u |\dot{\gamma}(t)| \, dt = \phi(u) ,$$

$$\dot{\phi}(u) = |\dot{\gamma}(u)| \neq 0 \Rightarrow \phi : I \to \operatorname{im} \phi =: \tilde{I}$$

is invertible, smooth. Inverse fn thm

$\Rightarrow \phi^{-1} : \tilde{I} \xrightarrow{\sim} I$ is smooth too.

Chain rule $\Rightarrow$ repar. $\tilde{\gamma} = \gamma \circ \phi^{-1} : \tilde{I} \twoheadrightarrow C$

has speed $|\dot{\tilde{\gamma}}(s)| = |\dot{\gamma}(u) \frac{du}{ds}| = 1$.

since $u = \phi^{-1}(s)$ $\square$

Sometimes call $s$ above the arc length parameter.