Lecture 17: Manifolds

Aim Intro. a more natural & flexible notion of n-folds.

Motivation

Some inadequacies of current defn of an n-fold $\mu: U \twoheadrightarrow M \subseteq \mathbb{R}^m$

* Often choice of par. irrelevant.
* Can't study whole sphere S^2.
* Ambient $\mathbb{R}^m$ shouldn't be nec. Want intrinsic view.

Manifold

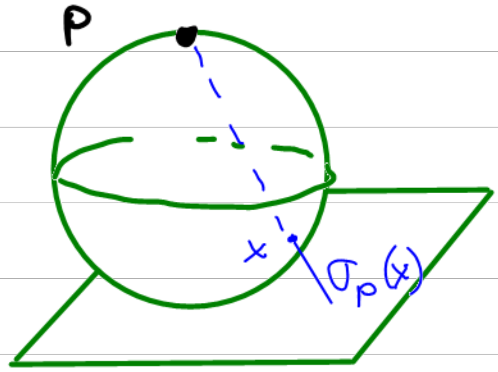
Defn 1 (Though we won't use this much) Our top spaces should be second countable ie. have a countable basis

Eg. 1. $\mathbb{R}^n$ has a countable basis of balls centred on $u \in \mathbb{Q}^n$ with rational radius.

Defn 2 An n-dimensional topological manifold (or topological n-fold) consists of a Hausdorff second countable top. space M s.t. every $p \in M$ has an open nbhd homeo to an open subset of $\mathbb{R}^n$. essentially locally Euclidean.

Eg. 2 The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ is

a top. n-fold :: $S^n - p \overset{ex}{\cong} \mathbb{R}^n$ via stereographic projn σ_p



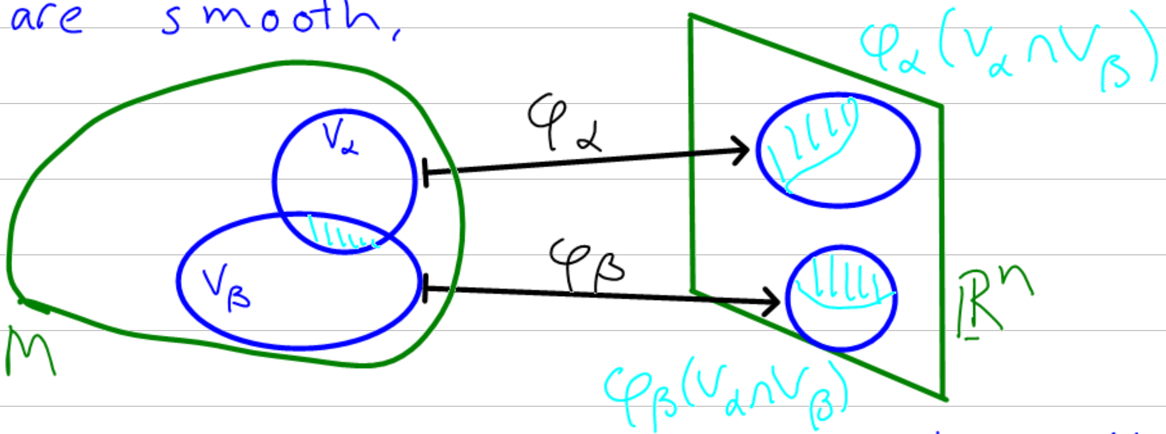
Defn 3 A smooth structure on a top n-fold M consists of conn. open subsets $V_\alpha \subseteq M$, $\alpha \in A$, and maps $\varphi_\alpha : V_\alpha \longrightarrow \mathbb{R}^n$ which are homeomorphisms onto open subsets of $\mathbb{R}^n$ (called charts) st.

a. $\bigcup V_\alpha = M$

b. For $\alpha, \beta \in A$, transition functions

$$\varphi_{\alpha\beta} : \varphi_\alpha(V_\alpha \cap V_\beta) \xrightarrow{\varphi_\alpha^{-1}} V_\alpha \cap V_\beta \xrightarrow{\varphi_\beta} \varphi_\beta(V_\alpha \cap V_\beta)$$

are smooth.



Later abuse terminology & say $\varphi_\beta \circ \varphi_\alpha^{-1}$ smooth.

We say M equipped with a smooth structure is a (smooth) manifold or n-fold. Denoted (M, φ_α) or M. The dimension of M is n.

$\boxed{\text{Examples}}$

Eg. 1. $\mathbb{R}^n$ with single chart $\varphi = id$.

Eg. 2 $\mathbb{C}^n$ with single chart $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$

defined by $\varphi(z_1,..,z_n)=(\mathrm{Re}\,z_1, \mathrm{Im}\,z_1, \mathrm{Re}\,z_2, \mathrm{Im}\,z_2, \dots)$

Eg.3. An open subset U of a manifold M with charts $\varphi_\alpha|_U$ is a manifold called an open submanifold.

Eg.4. Sphere S^n, with two charts σ_p, σ_q stereo. proj from distinct $p, q \in S^n$.

ex. Check $\sigma_q \sigma_p^{-1}$ smooth. see Problem Sets

Eg5 Product manifolds.

Consider manifolds M, charts $\varphi_\alpha : V_\alpha \longrightarrow \mathbb{R}^n$

$\tilde{M}$, charts $\tilde{\varphi}_{\tilde{\alpha}} : \tilde{V}_{\tilde{\alpha}} \longrightarrow \mathbb{R}^m$.

Then $M \times \tilde{M}$ is an $(m+n)$-fold with charts

$\varphi_{\alpha,\tilde{\alpha}} : V_\alpha \times \tilde{V}_{\tilde{\alpha}} \xrightarrow{\varphi_\alpha \times \tilde{\varphi}_{\tilde{\alpha}}} \mathbb{R}^n \times \mathbb{R}^m \overset{\text{nat.}}{\cong} \mathbb{R}^{m+n}$.

Why? Note $\bigcup_{\alpha,\tilde{\alpha}} V_\alpha \times V_{\tilde{\alpha}} = M \times \tilde{M}$.

Trans. fns $\varphi_{\beta\tilde{\beta}} \varphi_{\alpha\tilde{\alpha}}^{-1} : \varphi_\alpha(V_\alpha \cap V_\beta) \times \varphi_{\tilde{\alpha}}(\tilde{V}_{\tilde{\alpha}} \cap \tilde{V}_{\tilde{\beta}}) \longrightarrow \mathbb{R}^n_x \times \mathbb{R}^m_y$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \varphi_\beta \varphi_\alpha^{-1}(x) \\ \varphi_{\tilde{\beta}} \varphi_{\tilde{\alpha}}^{-1}(y) \end{pmatrix}$$

smooth.

Eg.6. Eg.4&5 $\Rightarrow$ n-torus $\mathbb{T}^n := (S^1)^n$ is an n-fold.

$\boxed{\text{Smooth Maps}}$

Prop-Defn Let $(M, \varphi_\alpha : V_\alpha \to \mathbb{R}^m)$, $(N, \psi_\beta : W_\beta \to \mathbb{R}^n)$

manifolds & $f: M \to N$ a fn. We say f is smooth at $p \in M$ if for some $V_\alpha \ni p$, $W_\beta \ni f(p)$

$\ast$ $\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(V_\alpha \cap f^{-1} W_\beta) \to V_\alpha \cap f^{-1} W_\beta \to W_\beta \to \mathbb{R}^n$

is smooth at $\varphi_\alpha(p)$.

In this case $\ast$ holds $\forall\, V_\alpha \ni p, W_\beta \ni f(p)$.

Proof: If $\ast$ holds & $p \in V_\gamma$, $f(p) \in W_\delta$ then locally at $\varphi_\gamma(p)$

$$\psi_\delta \circ f \circ \varphi_\gamma^{-1} = \underbrace{\psi_\delta \psi_\beta^{-1}} \psi_\beta f \varphi_\alpha^{-1} \underbrace{\varphi_\alpha \varphi_\gamma^{-1}}$$

smooth transition fn

is smooth being composite of smooth. $\square$

Defn 4 Let $f: M \to N$ be a fn between manifolds.

a f is smooth if it's smooth at every $p \in M$. Applies to fn $f: M \to \mathbb{R}$

b f is a diffeomorphism if it's smooth with a smooth inverse.

c f is a local diffeomorphism at $p \in M$, if it restricts to diffeo. between open nbhds of p & $f(p)$

d f is a local diffeomorphism if it's a local diffeo. at all $p \in M$ e.g. any chart $\varphi_\alpha : V_\alpha \to \mathbb{R}^n$.

e Two smooth str. $\{\varphi_\alpha\}, \{\psi_\beta\}$ on top. n-fold define the same manifold if $id : (M, \varphi_\alpha) \to (M, \psi_\beta)$ is a diffeo.

Eg. $(S^n, \{\sigma_p, \sigma_q\})$, $(S^n, \{\sigma_{p'}, \sigma_{q'}\})$ define the same manifold. see PS