Lecture 16: Local Gauss-Bonnet Theorem II

Aim Prove local Gauss-Bonnet for embedded par. surface $\sigma : U \xrightarrow{\sim} S \subset \mathbb{R}^3$ & polygon $P \subset U$ on it.

Turning Tangents & κ_g

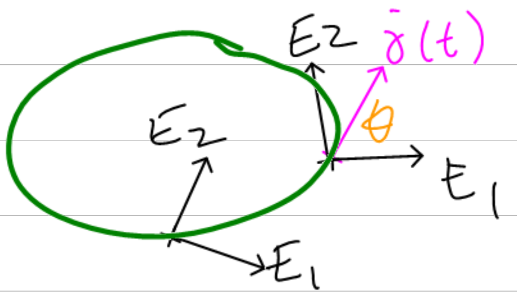
For now, only prove G-B thm in easy case where P suff. small that we shrink U & ass

I Coords are geodesic, $G = \begin{pmatrix} 1 & 0 \\ 0 & G_{22} \end{pmatrix}$.

II $\exists u \in U$ s.t. orthog. projn of S onto $T_u S$ is an homeo. Note this holds in a nbhd of every $u \in U$.

Consider o/n vector fields $E_1 = \partial_1 \sigma$, $E_2 = G_{22}^{-1/2} \partial_2 \sigma$. Let $\gamma = \sigma \circ c : I \to \sigma(\partial P)$ be unit speed par. of $\sigma(\partial P)$, orientn induced from orient. of σ.

Write $c(t) = (u^1(t), u^2(t))^T$.



Let $\varepsilon_1 = \dot\gamma$, $\varepsilon_2 = \nu \times \dot\gamma$ be osculating frame for γ.

Part ⓐ of next result follows as in lemma Lect 3

Prop ⓐ There's a p-wise smooth fn $\theta : I \to \mathbb{R}$ s.t.

$$\varepsilon_1(t) = \cos\theta(t)\, E_1(c(t)) + \sin\theta(t)\, E_2(c(t)).$$

(b) $\kappa_g(t) = \dot\theta(t) + \partial_1\sqrt{G_{22}}\,\dot{u}^2$.

Here u^1, u^2 are coord. for $U \subseteq \mathbb{R}^2_{u^1,u^2}$

Proof (b) $\langle E_i, E_i\rangle = 1 \overset{ex}{\Rightarrow} \langle \nabla_t E_i, E_i\rangle \overset{*}{=} 0$.

$\langle E_1, E_2\rangle = 0 \Longrightarrow \langle \nabla_t E_2, E_1\rangle \overset{*}{=} -\langle \nabla_t E_1, E_2\rangle$.

Now $\varepsilon_2 = -\sin\theta\, E_1 + \cos\theta\, E_2$ so

$\kappa_g(t) = \langle \nabla_t\, \varepsilon_1, \varepsilon_2\rangle$ essentially by defn.

$$\overset{\text{product rule}}{=} \langle \underbrace{-\dot\theta \sin\theta\, E_1 + \dot\theta \cos\theta\, E_2}_{=\dot\theta\,\varepsilon_2},\ \varepsilon_2\rangle$$

$$+ \langle \cos\theta\, \nabla_t E_1 + \sin\theta\, \nabla_t E_2,\ \varepsilon_2\rangle$$

$$\overset{\text{use }*}{=} \dot\theta + \langle \nabla_t \underset{\partial_1\sigma}{E_1}, \underset{G_{22}^{-1/2}\partial_2\sigma}{E_2}\rangle$$

$$= \dot\theta + \langle \sum_{k,j} \dot{u}^j \Gamma^k_{1j}\, \partial_k\sigma,\ G_{22}^{-1/2}\, \partial_2\sigma\rangle \quad \text{by Formula 1 Lect 14}$$

$$= \dot\theta + G_{22}^{1/2} \sum_j \dot{u}^j \Gamma^2_{1j}$$

Saw in proof Thm Lect 15 that $\Gamma^2_{11} = 0$

so suff prove $\Gamma^2_{12} = \partial_1\sqrt{G_{22}}/\sqrt{G_{22}}$.

Now Thm 1, Lect 12 $\Rightarrow$

$$\Gamma^2_{12} = \tfrac12 \sum G^{2l}\left(\partial_1 G_{2l} + \partial_2 G_{1l} - \partial_l G_{12}\right)$$

$$= \tfrac12 G^{22}\partial_1 G_{22}$$

$$= \tfrac12 \partial_1 G_{22}/G_{22} = \partial_1\sqrt{G_{22}}/\sqrt{G_{22}}$$

$\square$

$\boxed{\text{Proof Local Gauss-Bonnet Thm}}$

Thm For ext. angles α_j of P

$$\iint_{\sigma P} K_{Gauss}\, dS + \int_{\sigma(\partial P)} K_g\, ds + \sum \alpha_j = 2\pi$$

Prop $\Rightarrow$ this follows from

u^1, u^2 are coords on U.

Thm A $\displaystyle\iint_{\sigma P} K_{Gauss}\, dS + \int_{\partial P} \partial_1 \sqrt{G_{22}}\, du^2 = 0$

& Thm B $\displaystyle\int_{\partial P} \dot{\theta}\, dt + \sum \alpha_j = 2\pi$.

Proof Thm A. Prop Lect. 12 $\Rightarrow$

$$\iint_{\sigma P} K_{Gauss}\, dS = \iint_P -\frac{\partial_1^2 \sqrt{G_{22}}}{\sqrt{G_{22}}} \underbrace{|\partial_1\sigma \times \partial_2\sigma|}_{=\ E_1 \times \sqrt{G_{22}}\, E_2} du^1 du^2$$

$$= \iint_P - \underbrace{\partial_1^2 \sqrt{G_{22}}}_{=\ \mathrm{curl}\left(\partial_1 \sqrt{G_{22}}\, e_2\right)} du^1 du^2$$

standard basis vector in $\mathbb{R}^2$.

$$\underset{\text{Green's thm}}{=} - \int_{\partial P} \partial_1 \sqrt{G_{22}}\, du^2 \qquad \square$$

Proof Thm B (FYI Not examinable)

Umlaufsatz is the case of plane curve.

We continuously flatten out S onto the tangent plane T_uS, ie. for $\tau \in [0,1]$ we can construct par. surface $\sigma_\tau : U \to \mathbb{R}^3$ s.t. $\sigma_1 = \sigma$, $\sigma_0 = \pi \circ \sigma$ where $\pi : \mathbb{R}^3 \longrightarrow T_uS$ is orthog. proj & f'n $U \times [0,1] \longrightarrow \mathbb{R}^3 : (u,\tau) \longmapsto \sigma_\tau(u)$ is cont.

For each σ_τ, we let $2\pi n_\tau$ be value of corresp. $\int_{\partial P} \dot\theta dt + \sum \alpha_j$. All n_τ are integers & is cont. in τ $\therefore$ const. Umlaufsatz $\Rightarrow n_0 = 1$ so $n_1 = 1$ giving Thm B. $\square$

To remove U small assumption need:

$\boxed{\text{Dividing Polygons}}$ For polygons $P = P' \cup P''$ as in picture. Local Gauss-Bonnet for P' & P'' $\Rightarrow$ local Gauss-Bonnet for P. Why?

$\alpha + \pi = \alpha' + \alpha''$.

Now

$$\iint_{\sigma P'} K_{Gauss}\, dS + \iint_{\sigma P''} K_{Gauss}\, dS \overset{1}{=} \iint_{\sigma P} K_{Gauss}\, dS$$

More details for this in Lecture 24.

$$\int_{\sigma(\partial P')} \kappa_g\, ds + \int_{\sigma(\partial P'')} \kappa_g\, ds \overset{2}{=} \int_{\sigma(\partial P)} \kappa_g\, ds$$

Above $\Rightarrow$

for ext. angles $\alpha_i, \alpha_j', \alpha_k''$ of P, P', P'' resp.

$$\sum \alpha_j' + \sum \alpha_k'' \overset{3}{=} \sum \alpha_i + 2\pi$$

Now sum 1, 2, 3 & use local Gauss-Bonnet on P', P'', to get local G-B on P.

$\boxed{\text{Removing assumption } U \text{ small}}$

More details on this Lect. 26-33.

Hypotheses I, II on U hold in a nbhd of every $u \in U$. Basic idea is to keep dividing P into ever smaller polygons P_i, until every P_i lies in such a U. Then applying local G-B on each of these P_i, we combine to obtain local G-B on P noting $\iint_P K_{Gauss}\, dS$, $\int_{\partial P} \kappa_g\, dt$ "additive" & how ext. angles change.