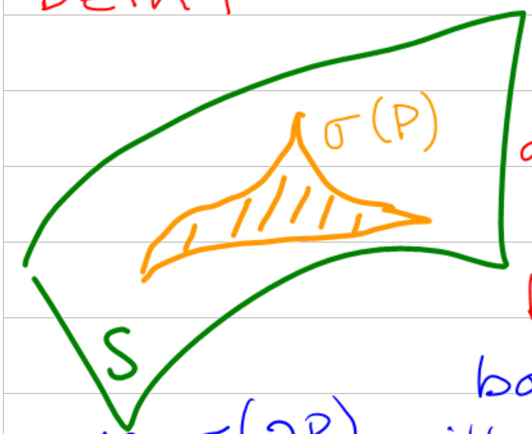
Lecture 15: Local Gauss-Bonnet Theorem I

Aim Generalise Hopf's Umlaufsatz from plane curves to certain curves on a par. surface $\sigma: U \twoheadrightarrow S$ embedded in $\mathbb{R}^3$.

Next two lectures are sketchier!!

Polygons

Defn 1



A polygon on S is a subset $P \subseteq U$ which

a is homeomorphic to unit disk $D = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$

b restricting σ to the boundary ∂P is a p-wise smooth curve $\sigma(\partial P)$ with ext. angles $< \pi$ ie. $\exists$ p-wise smooth par. curve $c: I \twoheadrightarrow \partial P$ with ext. angles $< \pi$.

N.B. Orientation on S induces orientation ↺ on $\sigma(\partial P)$ ie $\nu \times \dot{\gamma}$ points to interior of $\sigma(P)$.

We can also define exterior angles α_j at corners as in lect. 4 defn 3. Let κ_g be geodesic curvature of unit speed par. of $\sigma(\partial P)$.

Local Gauss-Bonnet Theorem

Thm Let P be a polygon on S with ext. angles α_j. Then

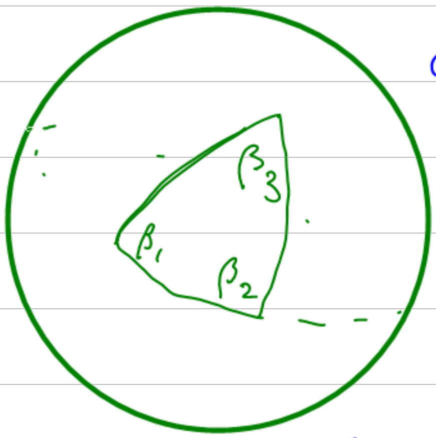
$$\iint_{\sigma P} K_{Gauss} \, dS + \underbrace{\int_{\sigma(\partial P)} \kappa_g \, ds}_{} + \sum \alpha_i = 2\pi.$$

s = arc length parameter so just integrating unit speed param.

Proof Next lecture but note for $S = \mathbb{R}^2$, $K_{Gauss} = 0$ so this is the Umlaufsatz. $\square$

Defn 2 A geodesic triangle (resp. n-gon) is a polygon P on S s.t. $\sigma|_{\partial P}$ consists of 3 (resp. n) geodesics.

E.g. 1 Let P be a geodesic triangle on unit sphere S^2 with interior angles $\beta_j = \pi - \alpha_j$.



$\sigma|_{\partial P}$ geodesic $\Rightarrow$ $\kappa_g = 0$, On unitsphere shape operator $L = \pm id$ by lect. 9 eg 2.

$\therefore K_{Gauss} = \det L = 1$.

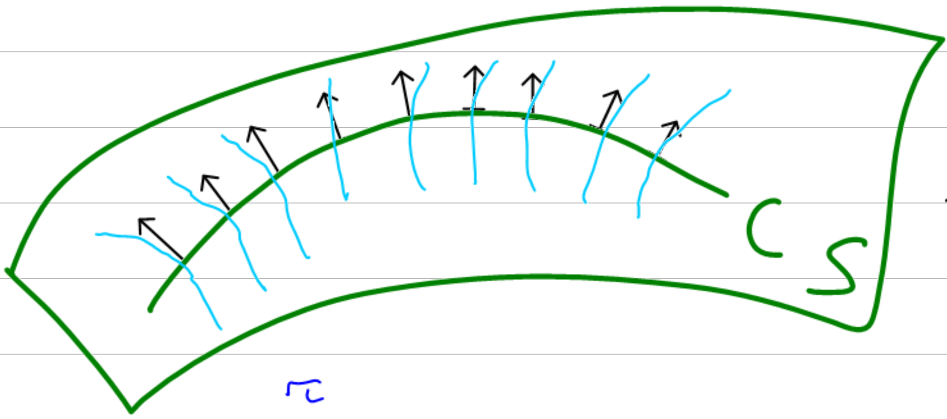
$$\text{Area } \sigma P = \iint_{\sigma P} K_{Gauss} \, dS$$

$$= 2\pi - \sum \alpha_j = \sum \beta_j - \pi.$$

Upshot Angle sum of spherical triangle is $> \pi$ & excess gives area!

$\boxed{\text{Geodesic Coordinates}}$ We'll use this important concept to simplify proof of THM.

geodesics



vector field $\perp C$.

Let $\gamma = \sigma \circ c : \hat{I} \to C \subset S$ be par. curve on S.

Consider geodesic $\gamma^\tau = \sigma \circ c^\tau : (-\varepsilon, \varepsilon) \to S$ with $\gamma^\tau(0) = \gamma(\tau)$, $\dot{\gamma}^\tau(0) = \dfrac{\nu \times \dot{\gamma}(\tau)}{|\dot{\gamma}(\tau)|}$.

Theory of ODEs $\Rightarrow$ for any $\tau_0 \in I$, on shrinking I to a smaller open nbhd, that we have a smooth map $\varphi : V := (-\varepsilon, \varepsilon) \times I \to U$ defined by

$$\varphi(v^1, v^2) = c^{v^2}(v^1).$$

Next result gives geodesic coord. in a suff. small nbhd of any point.

Thm Further shrinking I to a smaller nbhd of τ_0, & shrinking ε if nec., φ becomes a change of variables $\varphi : V \to \operatorname{im}\varphi$. The reparam. $\tilde{\sigma} = \sigma \circ \varphi : V \to S$ has geodesic coords i.e. 1st fund form $\tilde{G} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{G}_{22} \end{pmatrix}$.

Proof To show φ gives local change of var., suffice show $(d\varphi)_{(0,\tau_0)}$ invertible, or equiv. $(d\tilde{\sigma})_{(0,\tau_0)} : T_{(0,\tau_0)}\mathbb{R}^2 \to T_{(0,\tau_0)} S$ invertible

Along $v^1=0$: $\partial_2\tilde{\sigma} = \dfrac{d\gamma^\tau(0)}{d\tau} = \dot{\gamma}(\tau)$

At $(v^1,v^2)=(0,\tau)$: $\partial_1\tilde{\sigma} = \dfrac{d\gamma^\tau(t)}{dt}\Big|_{t=0} = \dfrac{\nu\times\dot{\gamma}(\tau)}{|\dot{\gamma}(\tau)|}$

These lin. indep. $\Rightarrow d\tilde{\sigma}_{(0,\tau_0)}$ invertible.

Now check coords geodesic. Note first curves $v^2=\text{const}$ are unit speed geodesics so $\tilde{G}_{11} = \langle \partial_1\tilde{\sigma}, \partial_1\tilde{\sigma}\rangle = 1$.

ODE for geodesics (lect. 14, Prop 2)

$0 = \ddot{v}^k + \sum \dot{v}^i\dot{v}^j\Gamma^k_{ij}$ for $v^1=t, v^2=\text{const}$

$k=1 \Rightarrow \Gamma^1_{11}=0$, $k=2 \Rightarrow \Gamma^2_{11}=0$.

$$0 = \Gamma^1_{11} = \tfrac{1}{2}\sum_l \tilde{G}^{1l}(2\partial_1\tilde{G}_{1l} - \partial_l\tilde{G}_{11}) = \tilde{G}^{12}\partial_1\tilde{G}_{12}.$$

$$= \tilde{G}^{12}\partial_1\tilde{G}_{12} = -(\det\tilde{G})^{-1}\tilde{G}_{12}\partial_1\tilde{G}_{12}$$

$$= -\tfrac{1}{2}(\det\tilde{G})^{-1}\partial_1(\tilde{G}_{12}^2)$$ square not index.

$\Rightarrow \tilde{G}_{12}$ const. wrt v_1.

But at $(0,\tau)$, $\partial_1\tilde{\sigma}\perp\partial_2\tilde{\sigma}$ so $\tilde{G}_{12}=0$

$\square$