Lecture 14: Geodesics

Aim Intro. analogues of "lines" on a par. surface $\sigma : U \twoheadrightarrow S \subset \mathbb{R}^3$.

Covariant Derivative

Defn 1 Let $\gamma = \sigma \circ c : I \to S$ be a curve on S. A vector field X along γ in S is a fn which maps $t \in I$ to $X(t) \in T_{\gamma(t)} S$ with smooth coord wrt $\partial_1\sigma, \partial_2\sigma$ i.e. $\exists X^i \in C^\infty$ s.t. $X(t) = X^1(t)(\partial_1\sigma)(\gamma(t)) + X^2(t)(\partial_2\sigma)(\gamma(t))$

For $u \in U$, let $\pi_u : T_{\sigma(u)}\mathbb{R}^3 \to T_u S$ be orthog. proj.

Defn 2 Let $\gamma = \sigma \circ c : I \to S$ be a par. curve on S & X a vector field along γ in S. View $X : I \to \mathbb{R}^3 : t \mapsto X(t) \in T_{\gamma(t)} S \subset \mathbb{R}^3$ so can define $\frac{dX}{dt}(t) \in T_{\gamma(t)}\mathbb{R}^3$. The covariant derivative of X at $t \in I$ is

$$\nabla_t X = \pi_{c(t)} \frac{dX}{dt} \in T_{\gamma(t)}\mathbb{R}^3.$$

This is a vector field along γ in S.

Express in terms of coords on U.

Suppose X is a vector field along γ in S so $X(t) = \sum_{i=1}^{2} X^i(t)\partial_i\sigma$. Write $c(t) = (u^i(t))$.

$$\frac{dX}{dt} = \sum_i \dot{X}^i \partial_i\sigma + \sum_{i,j} X^i(t)\partial_j\partial_i\sigma \, \dot{u}^j$$

$$= \sum_i \dot{X}^i \partial_i\sigma + \sum_{i,j,k} X^i \Gamma^k_{ij} \dot{u}^j \partial_k\sigma + \sum X^i H_{ij} \dot{u}^j \nu$$

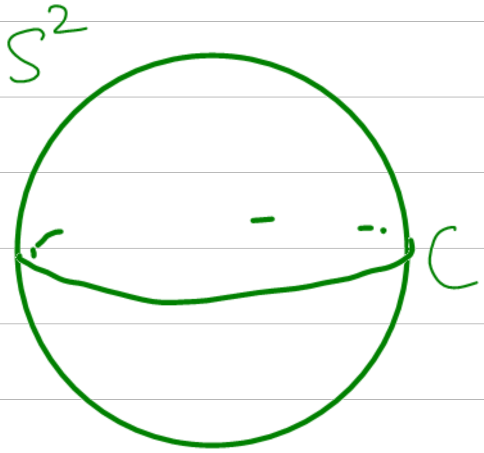
Formula 1 $\nabla_t X = \sum_i \dot{X}^i \partial_i \sigma + \sum_{i,j,k} X^i \dot{u}^j \Gamma^k_{ij} \partial_k \sigma$

$\boxed{\text{Geodesics}}$ Analogues of lines on surfaces are

Defn 3 Let $\gamma : I \to S$ be a curve on surface σ so $\dot{\gamma}(t), t \in I$ is a vector field along γ in S. We say γ is a geodesic if $\nabla_t \dot{\gamma} = 0$.

N.B This is indep. of par. of S but not γ!

Eg. 1 Unit speed param of great circles on spheres.



$\gamma = (\cos t, \sin t, 0)^T$ on unit sphere S^2.
This is a geodesic on S^2.
$\dot{\gamma}(t) = (-\sin t, \cos t, 0)^T$

$$\frac{d\dot{\gamma}}{dt} = \ddot{\gamma}(t) = (-\cos t, -\sin t, 0)^T = \pm \nu \perp T_{\gamma(t)} S^2$$

$\therefore \nabla_t \dot{\gamma} = 0$ & γ is a geodesic.

Physical Interpretation The trajectory γ is a geodesic iff the only net force is normal to the surface e.g. you are constrained to move on surface.

Prop 1 A curve γ on σ is a geodesic iff it is a constant speed curve with geodesic curvature $\kappa_g = 0$.

Proof Consider orthog. basis $\dot\gamma, \nu\times\dot\gamma, \nu$ at $T_{\gamma(t)}\mathbb{R}^3$. Now γ is a geodesic iff

a $\langle \ddot\gamma, \nu\times\dot\gamma\rangle = 0$ b $\langle \ddot\gamma, \dot\gamma\rangle = 0$

Now $\langle \ddot\gamma, \dot\gamma\rangle = \frac{1}{2}\frac{d}{dt}\langle \dot\gamma, \dot\gamma\rangle$ so

b $\iff$ speed $\langle \dot\gamma, \dot\gamma\rangle^{1/2}$ is const.

& a $\iff$ $\kappa_g = 0$ □

$\boxed{\text{Existence / Uniqueness}}$

First give intrinsic reformulation of geodesics in terms of coords.

Formula 1 $\Rightarrow$

Prop 2 Let $\gamma = \sigma\circ c : I \to S$ be a curve on σ, Write $c(t) = (u^1(t), u^2(t))^T$ so

$\dot\gamma(t) = \sum_k \frac{\partial\sigma}{\partial u^k}\cdot\frac{du^k}{dt} = \sum \dot u^k \partial_k\sigma$ by chain rule.

Then γ is a geodesic iff the following ODE holds

$$* \quad \ddot u^k + \sum_{i,j} \dot u^i \dot u^j \Gamma^k_{ij} = 0 \qquad k = 1, 2.$$

Rem We later can use $*$ to define geodesics when we don't have an embedding in $\mathbb{R}^3$, but only G & hence Γ^k_{ij}.

The theory of ODEs gives existence & uniqueness of solns to *

Thm For any $u \in U$, $v \in T_u S$, there is a geodesic $\gamma : (-\varepsilon, \varepsilon) \longrightarrow S$ with $\gamma(0) = u$, $\dot\gamma(0) = v$. It's unique in sense that another geodesic $\tilde\gamma : (-\tilde\varepsilon, \tilde\varepsilon) \longrightarrow S$ has $\gamma = \tilde\gamma$ on $(-\varepsilon, \varepsilon) \cap (-\tilde\varepsilon, \tilde\varepsilon)$.

E.g.1 again The only geodesics on a sphere are const. speed par. great circles or portions thereof.

Geodesics & Distances

Unfortunately we won't prove or make precise

Fact A geodesic γ passing thru $\gamma(a) = p$, $\gamma(b) = q$ min. lengths of "curves in a nbhd of γ" from p to q.

Prop 3 Local isometries map geodesics to geodesics.

Proof Use Prop 2 to see geodesic condn is the same in both cases. $\square$

E.g.2. Cylinder is locally isom. to $\mathbb{R}^2$, Straight lines geodesic in $\mathbb{R}^2$.