# Lecture 13: Isometries, Bending Invariants

**Aim** Show how $k_{Gauss}$ explains why paper can bend into cylinders but not spheres.

## $\boxed{\text{Isometries}}$

**Defn** Given par. surfaces $\sigma : U \to S \subset \mathbb{R}^3$, $\tilde{\sigma} : \tilde{U} \to \tilde{S} \subset \mathbb{R}^3$, a (local) isometry is a (local) diffeo $\varphi : U \to \tilde{U}$ s.t. $\varphi^* \tilde{\sigma}^* \tilde{G} = \sigma^* G$ where $G, \tilde{G}$ are 1st fund. forms of $\sigma, \tilde{\sigma}$.

**Rem.** Better defn later (Lect. 21) for (local) isometry $\varphi : S \to \tilde{S}$ of unparametrised surfaces via $\varphi^* \tilde{G} = G$.

**Eg 1** $\sigma = \tilde{\sigma} = id : \mathbb{R}^2 \to \mathbb{R}^2$.

$G = \tilde{G}$ $\quad G_{ij} = G(e_i, e_j) = \delta_{ij}$ ie.

$$(G_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \text{ identity matrix.}$$

Fix $v \in \mathbb{R}^2$ & rotation matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

so $A^T A = I$.

$\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : u \mapsto Au + v$ is an isometry

$\because$ $d\varphi = A$ & Lect. 8 Formula 1' $\overset{\text{check}}{\Longrightarrow}$

$$\underline{\varphi^* \tilde{G}} = (d\varphi)^T \tilde{G} (d\varphi) = A^T I A = I = \underline{G}.$$

**Prop 1** $\varphi : U \to \tilde{U}$ is a local isometry iff it preserves lengths of curves ie. given curve $\gamma = \sigma \circ c : I \to S$, let $\tilde{\gamma} = \tilde{\sigma} \circ \varphi \circ c : I \to \tilde{S}$; then length $\gamma$ = length $\tilde{\gamma}$.

Proof: $(\Rightarrow)$ length $\tilde{\gamma} = \int_I \sqrt{(\tilde{\sigma}^*\tilde{G})(d\varphi(\dot{c}(t)), d\varphi(\dot{c}(t)))}\,dt$

Note for $\tilde{c} = \phi \circ c$ have $\dot{\tilde{c}} = d\phi(\dot{c}(t))$ by chain rule.

$$= \int_I \sqrt{(\varphi^*\tilde{\sigma}^*\tilde{G})(\dot{c}(t), \dot{c}(t))}\,dt$$

$$= \int_I \sqrt{(\sigma^* G)(\dot{c}(t), \dot{c}(t))}\,dt$$

$$= \text{length } \gamma$$

$(\Leftarrow)$ Differentiating above equality shows $(\varphi^*\tilde{\sigma}^*\tilde{G})(\dot{c}(t), \dot{c}(t)) = (\sigma^* G)(\dot{c}(t), \dot{c}(t))$

But can find curve so $\dot{c}(t)$ is any tangent vector so $(\varphi^*\tilde{\sigma}^*G)(v,v) = (\sigma^*G)(v,v) \quad \forall v \in T_u S$.

Done $\because$ any symm. $g \in (\mathbb{R}^n)^{0,2}$ determined by values $g(v,v)$ since

$$g(v,w) = \tfrac{1}{2}[g(v+w, v+w) - g(v,v) - g(w,w)] \quad \square$$

## Cylinders

Eg 2 Let $\gamma(u) = (f(u), g(u))^T$ be a unit speed curve $\gamma: I \to C < \mathbb{R}^2$.

$\sigma(u,v) = (f(u), g(u), v)^T$ is a cylinder on $C$. $\sigma: I \times \mathbb{R} \to \mathbb{R}^3$

We compute $G$.

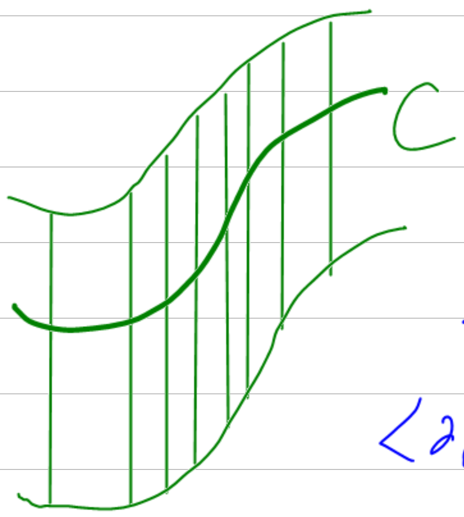
$$\partial_1 \sigma = (\dot{f}(u), \dot{g}(u), 0)^T$$

$$\partial_2 \sigma = (0, 0, 1)^T.$$

$\gamma$ unit speed $\Rightarrow$

$$G_{11} = \langle \partial_1\sigma, \partial_1\sigma \rangle = 1$$

$$\therefore \underbrace{(G_{ij})}_{\langle \partial_i\sigma, \partial_j\sigma \rangle} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\sigma^* G}$$



Consider portion of plane
$\tilde{\sigma}$ = inclusion: $I \times \mathbb{R} \longrightarrow \mathbb{R}^2$
which also has 1st fund. form
$\tilde{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by e.g. 1.
$\therefore$ $\varphi = id : I \times \mathbb{R} \circlearrowleft$ is an isometry

Physical Interpretation You can bend a flat sheet of paper $\tilde{\sigma}$ into cylinder $\sigma$ without stretching the paper at any pt.

## $\boxed{\text{Bending Invariants}}$

Theorema Egregium (Gauss) Let $\sigma : U \twoheadrightarrow S \subset \mathbb{R}^3$, $\tilde{\sigma} : \tilde{U} \twoheadrightarrow \tilde{S} \subset \mathbb{R}^3$ be two surfaces and $\varphi : U \to \tilde{U}$ be a local isometry at $u \in U$. Let $K_{Gauss}$, $\tilde{K}_{Gauss}$ be their respective Gaussian curvatures. Then

$$K_{Gauss}(u) = \tilde{K}_{Gauss}(\varphi(u)).$$

Defn 3 Such a quantity "preserved by local isometries" is called a bending invariant.

Proof: $\tilde{K}_{Gauss}$ is indep. of par. so can be computed using $\tilde{\tilde{\sigma}} = \tilde{\sigma} \circ \varphi : U \twoheadrightarrow \tilde{S}$.

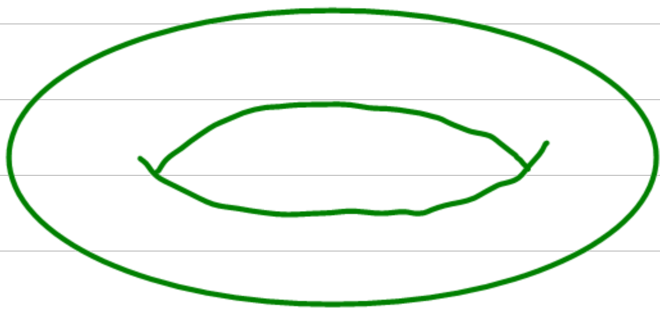
$\varphi$ a local isometry $\Rightarrow \tilde{\sigma}^* \tilde{G} = \sigma^* G$
But $K_{Gauss} = \frac{\det(H)}{\det(G)}$ is intrinsic (Thm 2 Lect 11)
so can be computed from the same quantity in both cases.

$\square$

**Scholium** Proof shows that you get bending invariants whenever you have an intrinsic quantity which is indep. of par.

**Physical E.g.** You can't bend a flat sheet of paper $\mathbb{R}^2$ onto a sphere $S^2$ $\because$ the Gaussian curvatures differ.

**E.g.** Isometries of the torus



$$\sigma : \mathbb{R}^2 \twoheadrightarrow S : (u,v)^T \longmapsto \begin{pmatrix} (a+b\cos u)\cos v \\ (a+b\cos u)\sin v \\ b\sin u \end{pmatrix}$$

Can compose rotations with reflections about $xy$-plane or $xz$-plane to get isometries. Note top & bottom parallels where $K=0$ are preserved or permuted.