Lecture 12: Christoffel Symbols

Aim Further study 2nd derivative quantities & relate G & H.

Let $\sigma : U \twoheadrightarrow S \subset \mathbb{R}^3$ be par. surface.

Non-orthogonal Fourier theory

Consider (0,2)-tensor g on V. (g_{ij}) coord. tensor wrt. basis $\{\varepsilon_i\}$.

Prop-Defn 1 Suppose that $\det g \neq 0$. Then there is an inverse (2,0)-tensor (g^{kl}), ie $n \times n$-array of numbers g^{kl}, $k,l = 1, \dots, n$ which satisfies $\sum_j g_{ij} g^{jk} = \delta_i^k$ & $\sum_j g^{ij} g_{jk} = \delta_k^i$

Proof Just use inverse of assoc matrix g. $\square$

E.g. $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g_{22} \end{pmatrix} \Rightarrow (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & g_{22}^{-1} \end{pmatrix}$.

Let $v = \sum_i v^i \varepsilon_i \in V$. Consider dual coordinates wrt g ie. coords (v_i) of $v^* := g(v,?) \in V^*$ wrt dual basis $\{\varepsilon^i\}$. So

ex $v_i = g(v, \varepsilon_i) = g(\sum v^j \varepsilon_j, \varepsilon_i) = \sum v^j g_{ji}$.

Prove this by showing $\sum g(v, \varepsilon_i) \varepsilon^i = g(v, ?)$

Formula 1 $v^k = \sum_i v_i g^{ik}$

Proof $\sum_i v_i g^{ik} = \sum_{i,j} v^j g_{ji} g^{ik}$

$$= \sum_{i,j} v^j \delta_j^k = v^k$$

$\square$

Rem In usual Fourier theory, $\{\varepsilon_i\}$ o/n $\Longleftrightarrow g_{ij} = \delta_j^i \Rightarrow g^{ij} = \delta_j^i$ & $v_i = v^i$.

Christoffel Symbols

For any $u \in U$, $T_u \mathbb{R}^3$ has basis $\partial_1\sigma, \partial_2\sigma, \nu$. $\therefore$ Can define smooth fns $\Gamma^k_{ij} : U \to \mathbb{R}$ by

$$\partial_i \partial_j \sigma = \sum_{k=1}^{2} \Gamma^k_{ij} \partial_k \sigma + H_{ij} \nu$$

Rem 1 Taking $\langle ?, \nu \rangle$ shows H_{ij} are coord. of 2nd fund. form.

Rem 2 $\Gamma^k_{ij} = \Gamma^k_{ji}$ $\because$ $\partial_i \partial_j = \partial_j \partial_i$

Defn 1 The $\Gamma^k_{ij}(u)$ are called Christoffel symbols.

Note $\det G > 0 \Rightarrow \exists$ inverse $(2,0)$-tensor (G^{ij})

Thm 1 $\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell} G^{k\ell} \left(\partial_i G_{j\ell} + \partial_j G_{i\ell} - \partial_\ell G_{ij} \right)$

Proof Formula 1 $\Rightarrow$ suff. show

$$\langle \partial_i \partial_j \sigma, \partial_\ell \sigma \rangle = \frac{1}{2} \left(\partial_i G_{j\ell} + \partial_j G_{i\ell} - \partial_\ell G_{ij} \right)$$

$$\partial_i G_{j\ell} \overset{1}{=} \partial_i \langle \partial_j \sigma, \partial_\ell \sigma \rangle = \langle \partial_i \partial_j \sigma, \partial_\ell \sigma \rangle + \langle \partial_j \sigma, \partial_i \partial_\ell \sigma \rangle$$

$$\partial_j G_{i\ell} \overset{2}{=} \langle \partial_j \partial_i \sigma, \partial_\ell \sigma \rangle + \langle \partial_i \sigma, \partial_j \partial_\ell \sigma \rangle$$

$$\partial_\ell G_{ij} \overset{3}{=} \langle \partial_\ell \partial_i \sigma, \partial_j \sigma \rangle + \langle \partial_i \sigma, \partial_\ell \partial_j \sigma \rangle$$

1 + 2 − 3 gives thm. $\square$

Rem 3 Thm 1 $\Rightarrow$ the Γ^k_{ij} are intrinsic i.e. defined in terms of the G_{ij} and its derivatives (of any order). We need not invoke H or equiv. L.

$\boxed{\text{Gauss Equation}}$

DO NOT ERASE

No need to memorise this formula.

Thm 2 (Gauss)

$$H_{ij}H_{ks} - H_{ik}H_{js} \overset{*}{=} \sum_l \partial_k \Gamma^l_{ij} G_{ls} + \sum_{l,t} \Gamma^t_{ij} \Gamma^l_{tk} G_{ls}$$

$$- \sum_l \partial_j \Gamma^l_{ik} G_{ls} - \sum_{l,t} \Gamma^t_{ik} \Gamma^l_{tj} G_{ls}$$

Rem ① This is intrinsic since the Γ^l_{ij} are.
② For $(i,j,k,s) = (1,1,2,2)$, get LHS $= \det(\underline{H})$.

Proof Comes from $\partial_k \partial_j \partial_i \sigma = \partial_j \partial_k \partial_i \sigma$ & taking $\langle ?, \partial_s \sigma \rangle$.

$$\partial_k \partial_j \partial_i \sigma = \partial_k\left(\sum \Gamma^l_{ij} \partial_l \sigma\right) + \partial_k(H_{ij} \nu)$$

$\therefore$

$$\langle \partial_k \partial_j \partial_i \sigma, \partial_s \sigma \rangle = \sum_l \partial_k \Gamma^l_{ij} G_{ls} + \sum \Gamma^l_{ij} \langle \partial_k \partial_l \sigma, \partial_s \sigma \rangle$$

$$+ \langle H_{ij} \partial_k \nu, \partial_s \sigma \rangle$$

$$= \sum \partial_k \Gamma^l_{ij} G_{ls} + \sum \Gamma^l_{ij} \Gamma^t_{lk} G_{ts} - H_{ij} H_{ks}.$$

This gives 3 of the terms in $*$.

The other 3 come from $\langle \partial_j \partial_k \partial_i \sigma, \partial_s \sigma \rangle$ by swapping j & k in above calculation.

$\square$

Defn 2 We say σ has **geodesic coordinates** if $(G_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & G_{22} \end{pmatrix}$. Explain terminology in later lecture.

2nd derivative wrt. u^1.

Prop If σ has geodesic coord. then
$$K_{Gauss} = -\left(\partial_1^2 \sqrt{G_{22}}\right) / \sqrt{G_{22}}.$$
Remember this formula

Proof Thm 1 $\Rightarrow \Gamma^k_{ij} = \frac{1}{2} \sum_l G^{kl} (\partial_i G_{jl} + \partial_j G_{il} - \partial_l G_{ij})$

Geodesic coord. $\Rightarrow \Gamma^k_{ij} = \frac{1}{2} G^{kk} (\partial_i G_{jk} + \partial_j G_{ik} - \partial_k G_{ij})$

$\therefore 0 = \Gamma^1_{11} = \Gamma^1_{12} = \Gamma^2_{11}$

(need two of the indices i, j, k equal to 2 to be non-zero)

Thm 2 $\Rightarrow$

$$\det(H_{ij}) = H_{11} H_{22} - H_{12} \underbrace{H_{21}}_{= H_{12}} = \cancel{\partial_2 \Gamma^2_{11} G_{22}} + \cancel{\sum_t \Gamma^t_{11} \Gamma^2_{t2} G_{22}}$$
$$- \partial_1 \Gamma^2_{12} G_{22} - \sum \Gamma^t_{12} \Gamma^2_{t1} G_{22}$$

$\det(G_{ij}) = G_{22}$ so

$$K_{Gauss} = \det(H_{ij}) / \det(G_{ij})$$
$$= -\partial_1 \Gamma^2_{12} - \Gamma^2_{12} \Gamma^2_{12}$$
$$= -\partial_1 \left(\frac{1}{2} \underbrace{G^{22}}_{= 1/G_{22}} \partial_1 G_{22} \right) - \left(\frac{1}{2} G^{22} \partial_1 G_{22} \right)^2$$

$$= -\frac{1}{2}\frac{\partial_1^2 G_{22}}{G_{22}} + \frac{(\partial_1 G_{22})^2}{2(G_{22})^2} - \frac{1}{4}\frac{(\partial_1 G_{22})^2}{(G_{22})^2}$$

$$= -\frac{1}{2}\frac{\partial_1^2 G_{22}}{G_{22}} + \frac{1}{4}\frac{(\partial_1 G_{22})^2}{(G_{22})^2}.$$

$$\partial_1 \sqrt{G_{22}} = \frac{\partial_1 G_{22}}{2\sqrt{G_{22}}}$$

$$\partial_1^2 \sqrt{G_{22}} = \frac{2\sqrt{G_{22}}\,\partial_1^2 G_{22} - (\partial_1 G_{22})^2/\sqrt{G_{22}}}{4G_{22}}$$

$\square$

Rem Proof of this Prop. depends on formula 1 $K_{Gauss} = \det(H_{ij}) / \det(G_{ij})$

& 2 Thm 2 formula for $\det(H_{ij})$.

Later we **define** K_{Gauss} this way so Prop. still holds.