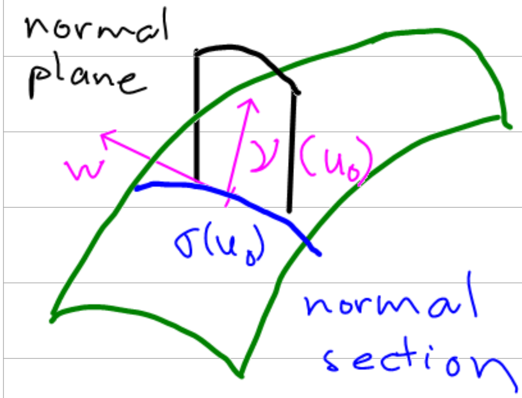
Lecture 10: Normal Curvature

Aim Interpret 2nd fund. form H on par. surface $\sigma : U \to S \subset \mathbb{R}^3$ via curvature of certain curves on S.

Normal Sections



Defn 1 The normal plane to S at u_0 in direction $w \in T_{u_0}S$, $w \neq 0$, is the plane $P \subset \mathbb{R}^3$ thru $\sigma(u_0)$ parallel to w & $\nu(u_0)$

A normal section to S at u_0 in direction w is a curve on S passing thru $\sigma(u_0)$ lying in P.

Prop 1 For any $w \in T_{u_0}S$, there exists a normal section at u_0 in dirn w.

Proof FYI.

Consider fn $\pi : \mathbb{R}^3_x \to \mathbb{R}^2$ defined by $\pi(x) = \begin{pmatrix} \langle w, x \rangle \\ \langle w \times \nu(u_0), x \rangle \end{pmatrix}$ essentially orthog. proj. onto $T_{u_0}S$

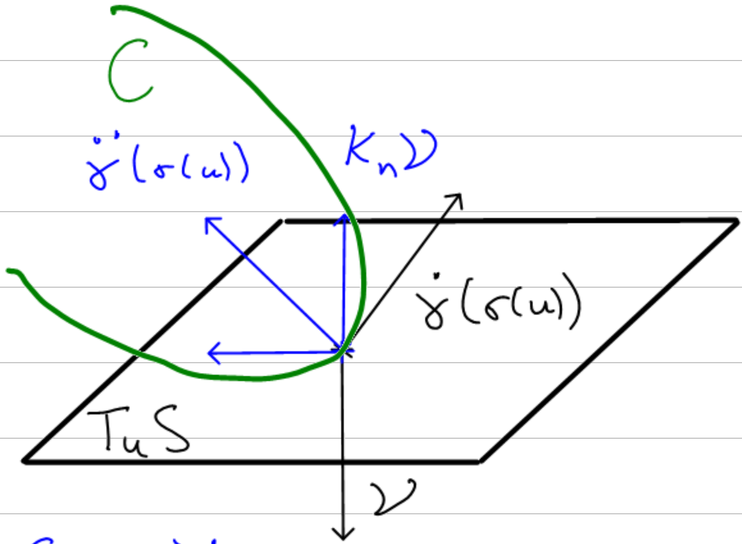
Let $\varphi = \pi \circ \sigma : U \to \mathbb{R}^2$.

$(d\varphi)_{u_0} = d\pi_{\sigma(u_0)} \circ d\sigma_u$ invertible $\because$

$d\pi|_{T_{u_0}S} : T_{u_0}S \to T_{\varphi(u_0)}\mathbb{R}^2$ is.

$\therefore$ On open nbhd V of $\varphi(u_0) = \binom{b}{c} \in \mathbb{R}^2$
we have a smooth inverse $\varphi^{-1}: V \longrightarrow U$.
Normal section is
$\gamma(t) = \sigma(\varphi^{-1}(t, c))$ which lies
in norma plane $\pi^{-1}(\mathbb{R} \times c)$ since
$\pi\gamma(t) = \pi\sigma(\varphi^{-1}(t,c)) = \varphi\varphi^{-1}(t,c) \subseteq \{(t,c)\}$. $\square$

Normal Curvature



Consider curve $\gamma = \sigma \circ c : I \longrightarrow C$ on S.
Ass. unit speed so $|\dot{\gamma}(t)| = 1$, $\ddot{\gamma}(t) \perp \dot{\gamma}(t)$.
Frenet frame $\varepsilon_1 = \dot{\gamma}$, $\varepsilon_2 = \ddot{\gamma}/|\ddot{\gamma}|$ so
its curvature $\kappa_c = \left\langle \ddot{\gamma}(t), \frac{\ddot{\gamma}(t)}{|\ddot{\gamma}(t)|} \right\rangle = |\ddot{\gamma}(t)|$.

Have RH orthog. basis $\dot{\gamma}(t), \nu(t) \times \dot{\gamma}(t), \nu(t)$

Defn 2 Write $\ddot{\gamma}(t) = \kappa_n \nu(t) + \kappa_g \nu(t) \times \dot{\gamma}(t)$.
The normal curvature of C at u is κ_n & its geodesic curvature is κ_g.

N.B. (a) $\kappa_c = \sqrt{\kappa_n^2 + \kappa_g^2}$

(b) A normal section C at u_0 has $\kappa_g(u_0) = 0$
so $\kappa_c(u_0) = |\kappa_n(u_0)|$

$\boxed{\text{Meusnier's Thm}}$

Note $H(u)(v,v)$ is a quadratic form on T_uS.

Thm Let $\gamma = \sigma \circ c : I \to C$ be a unit speed curve on S. Then $H(\dot\gamma(t), \dot\gamma(t)) = \langle \ddot\gamma(t), \nu(c(t)) \rangle = \kappa_n(t)$.

Proof Diff wrt t, $0 = \langle \dot\gamma(t), \nu(t) \rangle \Rightarrow$

$$\langle \ddot\gamma(t), \nu(t) \rangle = -\langle \dot\gamma(t), \dot\nu(t) \rangle$$

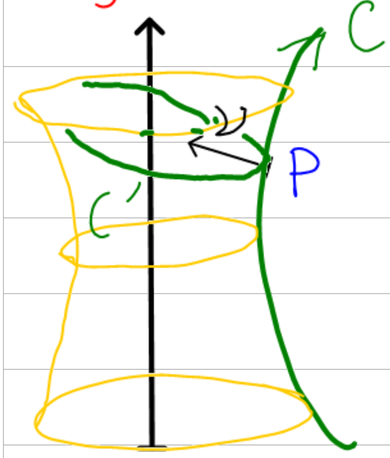
$$= -\langle \dot\gamma(t), d\nu \circ \underbrace{\dot c(t)}_{= (d\sigma)^{-1}(\dot\gamma(t))} \rangle = H(\dot\gamma(t), \dot\gamma(t))$$

abused notn for $\nu(c(t))$

$\square$

Cor For unit vector $w \in T_uS$, $|H(w,w)|$ is the curvature of a normal section in dirn w.

E.g. 1.



σ = surface of revolution of $C \subset \mathbb{R}^2_{x,z}$.

If C bends away from normal, its normal curvature $\kappa_{n,C} < 0$. But C' in picture has $\kappa_n > 0$.

P is a saddle point.

$\boxed{\text{Sign of determinant of (0,2)-tensor}}$

Let g be (0,2)-tensor on vector space

$V/\mathbb{R}$. Define the determinant of its coord. tensor (g_{ij}) wrt basis $\{\varepsilon_i\} \subset V$ to be

$$\det(g_{ij}) = \det \underline{g} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} g_{i\sigma(i)}.$$

Prop-Defn $\operatorname{sgn} \det \underline{g}$, defined to be the sign ± 1 or 0 of $\det(g_{ij})$ is indep. of choice of basis.

Proof If you change basis with $A \in GL_n(\mathbb{R})$, then coord. tensor has assoc. matrix (see lect. 8, Formula 1′)

$$\underline{\tilde{g}} = A^T \underline{g} A \Rightarrow \det(\underline{\tilde{g}}) = \det(A)^2 \det(\underline{g}).$$

$\square$

N.B $\det \underline{g}$ not defined but write $\det \underline{g} > 0$ if $\operatorname{sgn}\det \underline{g} = 1$ & $\det \underline{g} < 0$ if $\operatorname{sgn} \det \underline{g} = -1$.

Prop 2 The first fund. form G has $\det G > 0$.

Proof This follows since G is positive definite i.e. $G(v,v) \geq 0$ & $G(v,v) = 0 \Rightarrow v = 0$.

$\square$

Defn 3 σ is elliptic (resp. parabolic, hyperbolic) at $u \in U$ if $\det H(u) > 0$ (resp. $= 0$, < 0).

Eg. 1 again The saddle point is hyperbolic.