Lecture 1: Modern Geometry

Themes in this Course

* Curvature
* Manifolds
* Topology

Smooth

Co-ord. on $\mathbb{R}^n = \mathbb{R}^n_u = \mathbb{R}^n_{u^1,\dots,u^n}$
$u^1, \dots, u^n$.
If confusing & n small, rewrite
$u^1 \rightsquigarrow u$, $u^2 \rightsquigarrow v$, $u^3 \rightsquigarrow w$.
Sim. $\mathbb{R}^3_{x^1,x^2,x^3} \rightsquigarrow \mathbb{R}^3_{x,y,z}$.

Defn 1 Let $U \subseteq \mathbb{R}^n$ be open.
A fn $f: U \to \mathbb{R}$ is smooth if all partial derivatives of f of all orders (i.e. $\partial_i f = \frac{\partial f}{\partial u^i}$, $\frac{\partial^2 f}{\partial u^i \partial u^j}$, etc) exist. The vector space of these denoted $C^\infty(U)$.
A fn $f: U \to V \subseteq \mathbb{R}^m_x : u \longmapsto (f^1(u), \dots, f^m(u))^T$ is smooth if all co-ord. fns f^i are.

Defn 2 Let $U \subseteq \mathbb{R}^n_u$, $V \subseteq \mathbb{R}^m_x$ be open subsets. A smooth fn $f: U \to V$ is a diffeomorphism if it's invertible with f^{-1} smooth.

It's a local diffeomorphism at $u \in U$ if there's an open nbhd $U' \ni u$ s.t. $f : U' \to f(U')$ is a diffeo. It's a local diffeomorphism if it's a local diffeo at every $u \in U$.

Eg 1 a $f : \mathbb{R} \to \mathbb{R}_{>0} : x \mapsto e^x$ is a diffeo $\because$ inverse log smooth.

b $g(x) = x^2$ restricts to local diffeo. $g : \mathbb{R}-0 \to \mathbb{R}_{>0}$ $\because$ restricts to diffeo. on $\mathbb{R}_{>0}$, $\mathbb{R}_{<0}$ with smooth inverses $y \mapsto \pm\sqrt{y}$.

Tangent Spaces

$U \overset{\text{open}}{\subseteq} \mathbb{R}^n_u$

Defn 3 The tangent space at $p \in U$ is the copy of $\mathbb{R}^n$,

$$T_pU := p \times \mathbb{R}^n \subseteq U \times \mathbb{R}^n.$$

Typically view $(p, v) \in T_pU$ as arrow tail $p \mapsto p + v =$ head.

Annoyingly oft identify T_pU with $\mathbb{R}^n$ giving it vector space str. with std basis vectors e_i also denoted $\partial_i = \frac{\partial}{\partial u^i}$.

$\boxed{\text{Derivatives}}$

$U \overset{open}{\subseteq} \mathbb{R}^n_u$, $X \overset{open}{\subseteq} \mathbb{R}^m_x$

Let $f : U \to X$ be smooth. Recall for $u \in U$ have Jacobian matrix $J(u)^i_j$ with (i,j)-th entry $J(u)^i_j = \frac{\partial f^i}{\partial u^j}(u) = \frac{\partial x^i}{\partial u^j}(u)$

tensor notn for matrices.

It represents linear derivative map

Recall gives best linear approx.

$$df_u : T_u U \longrightarrow T_{f(u)} X$$

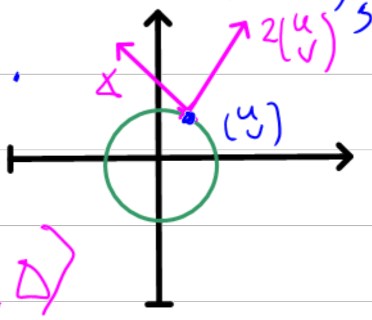
$$\sum \alpha^j \underset{\frac{\partial}{\partial u^j}}{\underset{\shortparallel}{e_j}} \longmapsto \sum_{i,j} J(u)^i_j \, \alpha^j \underset{\frac{\partial}{\partial x^i}}{\underset{\shortparallel}{e_i}}.$$

e.g. $f(u,v) = u^2 + v^2 \Rightarrow df_{(u,v)} : T_{(u,v)}\mathbb{R}^2 = \mathbb{R}^2 \to T_{f(u,v)}\mathbb{R}$

is left mult^n by $(2u \;\; 2v) = \left\langle \binom{2u}{2v}, ? \right\rangle$

dot product with $\binom{2u}{2v}$.

our notn for dot product.



$\Delta \in T_{(u,v)}\mathbb{R}^2$

$f((u,v) + \Delta) \simeq f(u,v) + \left\langle 2\binom{u}{v}, \Delta \right\rangle$

$\boxed{\text{Parametrised } n\text{-folds}}$

Objects of study in this course are the following and their variants.

Defn 4 A parametrised n-fold in $\mathbb{R}^m$ is a conn. open $U \subseteq \mathbb{R}^n_u$ and surj. smooth map $\mu : U \twoheadrightarrow M \subseteq \mathbb{R}^m$ s.t. $d\mu_p$ is inj.

notn for onto.

for all $p \in U$.

It's embedded if μ is an homeo. onto its image.