Non-commutative Projective Geometry

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joint work with Adam Nyman

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Objects of Study

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Need to review some basic commutative algebraic geometry to make sense of this question properly.

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There's a surjective map $\mathbb{P}_X(V) \xrightarrow{f} X$ with fibre $f^{-1}(x) = \mathbb{P}(V_x)$.

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Key Point The defn of point modules makes sense for nc graded algebras too. In fact, Artin-Tate-Van den Bergh (1990) use them crucially in their study of 3-dim Sklyanin algebras to prove the aforementioned facts about them.

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Mori's philosophy for studying higher dimensional varieties

Let Y = smooth projective scheme (think manifold)
Try to construct map $f : Y \longrightarrow X$ with X and the fibres of f simpler than Y.
How? Consider the important invariant, the dualising sheaf

$$\omega_Y := \bigwedge^{\text{top}} T_Y^*,$$

which is a line bundle on Y.

Look for K-negative curves on Y, i.e. curves $C \subset Y$ s.t.

$$0 > K.C := \deg \omega_Y |_C = \deg \mathcal{O}_C \otimes_Y \omega_Y$$
$$= \text{“no. zeros of a global section of } \omega_Y |_C\text{”}.$$

If they exist, then for an extremal K-negative curve C, $\exists f : Y \longrightarrow X$ which contracts a curve $C' \subset Y$ iff C, C' are proportional in $H_2(Y, \mathbb{R})$.
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Thm

Let Y be a comm smooth proj surface and $C \subset Y$ an extremal K-negative curve with $C^2 = 0$, then there's a morphism $f : Y \longrightarrow X$ to a smooth proj curve X, contracting C, exhibiting Y as a $\mathbb{P}^1$-bundle. In particular, $C \simeq \mathbb{P}^1$ so $H^1(\mathcal{O}_C) = 0$.

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- In 1995, Artin-Stafford showed non-commutative smooth projective curves were essentially commutative. Thus non-commutative surfaces are the focus of research now.
- In 1997, Artin conjectured that, "up to birational equivalence", a non-commutative projective surface is either a nc $\mathbb{P}^1$-bundle over a curve or finite over its centre.

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E.g. Proj $Skl(a,b,c)$ is a nc $\mathbb{P}^2$.
E.g. Proj $Skl_4/(t)$ for generic $t \in Z(Skl_4)_2$ is a nc quadric or $\mathbb{P}^1 \times \mathbb{P}^1$.

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$$\operatorname{Ext}^i_Y(M, N) \simeq \operatorname{Ext}^{d-i}_Y(N, M \otimes \omega_Y)^*$$

natural in $M, N \in \operatorname{mod} Y, \omega_A = \omega_Y[d+1]$.

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On $Y = \text{Proj } A$, we can sensibly define

$$\dim = gk - 1 \quad \text{or} \quad c.\dim -1$$

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Intersection theory (I.Mori-Smith)

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$$f^* : \text{Mod}\, X \longrightarrow \text{Mod}\, Y, f_* : \text{Mod}\, Y \longrightarrow \text{Mod}\, X.$$

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Theorem (C-Nyman)

- *X is a projective curve which is smooth at the point p corresponding to M.*
- *If for every simple 0-dim quotient $P \in \operatorname{mod} Y$ of M we have $M.P = 0$, then there is a morphism $f : Y \longrightarrow X$ and $f^* \mathcal{O}_p = M$.*

Non-commutative Mori contraction

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M = K-non-eff rat curve with $M^2 = 0$,
X = comp of Hilb $\mathcal{O}_Y$ containing M.

Theorem (C-Nyman)

- *X is a projective curve which is smooth at the point p corresponding to M.*
- *If for every simple 0-dim quotient $P \in \text{mod}\, Y$ of M we have $M.P = 0$, then there is a morphism $f : Y \longrightarrow X$ and $f^* \mathcal{O}_p = M$.*

Idea of proof. Associated to X is a universal flat family $\mathcal{M} \in \text{mod}\, Y_X$ of quotients of $\mathcal{O}_Y$.

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$$f^* : \text{mod}\, X \xrightarrow{\mathcal{M} \otimes_X -} \text{mod}\, Y_X \xrightarrow{\pi_*} \text{mod}\, Y.$$

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One has to check this is right exact.