Singularities in the Mori program for orders

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Introduction

always work over $k = \mathbb{C}$

Object of Study We study "normal" orders on surfaces

1. in terms of geom data *ramification*
2. by analogy with comm alg geometry

This talk Give overview of Mori program for orders & see how McKay correspondence & matrix factorisation theory pan out in this setting.

Today work on surface = noetherian excellent 2-dim scheme
with res fields at closed pts k.
e.g. Spec R for R 2-dim complete local noeth res field k.

Throughout let Z = normal surface.

Normal orders

Let $A =$ sheaf of $\mathcal{O}_Z$-algebras

Defn

A is an *order* on Z if

- A is coherent & torsion-free as a sheaf
- $k(A) := A \otimes_Z k(Z)$ is a central simple k(Z)-algebra

Defn

An order A is *normal* if

- A is reflexive as a sheaf
- For every irred curve C, $\text{rad}(A \otimes_Z \mathcal{O}_{Z,C})$ is gen by a single (nec normal) elt called a *uniformiser* (so $A \otimes_Z \mathcal{O}_{Z,C}$ is hereditary).

Fact A maximal $\Longrightarrow$ normal $\Longrightarrow$ tame

e.g. For $\zeta = \sqrt[e]{1}$, skew power series ring $k_\zeta[[x, y]] = k\langle\langle x, y\rangle\rangle/(yx - \zeta\, xy)$ is a maximal order over $k[[u = x^e, v = y^e]]$.

Primary Ramification

A = normal order on Z
Note A is generically Azumaya.
Let C = ramification curve i.e.
$A_{Z,C} := A \otimes_Z \mathcal{O}_{Z,C}$ is not Azumaya. Let π = uniformiser.

Classical Fact

$$Z(A_{Z,C}/\text{rad}A_{Z,C}) = K^n$$

for some cyclic field ext $K/k(C)$. Further, Galois action induced by conjugation by π.

Measure failure of Azumaya by

Defn

The *ramification index* of A at C is

$$e_C := \deg K^n/k(C).$$

Secondary ramification

Ramification of cyclic field ext $K/k(C)$ gives secondary ramification.
e.g. $A = k_\zeta[[x,y]], \zeta = \sqrt[e]{1}$, $Z =$ Spec $k[[u = x^e, v = y^e]]$.

- Let C_u be curve $u = 0$, C_v be sim etc
- A ramified only on C_u, C_v.
- For $C = C_u$, $A_{Z,C}/\text{rad}A_{Z,C} = A/(x) = k((y))$.
- (Primary) ram index $e_C = \deg k((y))/k((v)) = e$
- Secondary ram index is also e.

Modifications

Rem In comm alg geom, study singularities by considering modifications e.g. blowups.

Setup Let $f : Z' \longrightarrow Z$ be a modification i.e.
proj birational morphism of normal surfaces.
Let $A =$ normal order on Z.

Defn

"The" *modification* of A wrt f is the normal order $f^{\#}A$ on Z' defined locally at irred curve C by

- $(f^{\#}A)_{Z',C} = (f^{*}A)_{Z',C} = A_{Z,f(C)}$ if C not exc.
- $(f^{\#}A)_{Z',C} =$ max order containing $f^{*}A_{Z',C}$ if C exc.

Rem Ram indices of $f^{\#}A$ at smooth rat exc curves is determined by 2ndary ram data.

Canonical divisor

Rem Key invariant in comm alg geom is canonical divisor.
Let $A =$ normal order on Z

Defn

Define the *canonical divisor* of A to be

$$K_A = K_Z + \sum_C (1 - \frac{1}{e_C}) C \in \text{Div} Z$$

where $e_C =$ ram index of A at C.

Motivation $\omega_A^{\otimes n} = A \otimes_Z \mathcal{O}(nK_A)$ in codim 1 for n suff large & divisible.
Suggests we define associated log surface

$$\text{Log}(A) = (Z, \Delta_A = \sum (1 - \frac{1}{e_C}) C)$$

Rem This retains only primary ram data.

Discrepancy

Rem Classes of sing in comm Mori program defined by how K changes wrt modifications.
Let $A =$ normal order on Z
For any modification $f : Z' \longrightarrow Z$ with exc curves $\{E_i\}$ we write

$$K_{f^\# A} \equiv f^* K_A + \sum_i a_i E_i$$

We define the *discrepancy* of A to be $\mathrm{disc}(A) = \inf\{e_i a_i\}$ where e_i is ram index of $f^\# A$ at E_i & infimum is over all modifications.

Defn

We say A is *terminal, canonical, log terminal* if $\mathrm{disc}(A) > 0, \geq 0, > -1$ respectively.

Surprise This is an interesting and useful definition.

Terminal orders

Rem For comm surfaces, terminal = smooth.

Theorem (C.-Ingalls 2005, Smoothness)

Any terminal order locally has finite global dimension.

Theorem (C.-Ingalls 2005, local structure of ramification)

An $\mathcal{O}_Z$-order is terminal iff Z is smooth and

- *the union of ram curves only has ordinary nodes as sing &*
- *the 2ndary ram index at any node = ram index of one of the ram curves passing through it.*

Theorem (C.-Ingalls 2005, Resolution of singularities)

For any normal order A on Z, there is a unique minimal modification $f : Z' \longrightarrow Z$ s.t. $f^{\#}A$ is terminal.

Local algebraic structure of terminal orders

From now on, R denotes a comm complete local noeth normal domain with residue field k.
Let $\zeta = \sqrt[e]{1}$ and $A(e) = k_\zeta[[x, y]]$.
Define

$$A(n,e) = \begin{pmatrix} A(e) & A(e) & \dots & A(e) \\ (x) & A(e) & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ (x) & \dots & (x) & A(e) \end{pmatrix} \subseteq A(e)^{n \times n}$$

Fact $A(n, e)$ is a terminal order with centre $k[[u = x^e, v = y^e]]$ & ram curves C_u, C_v with ram indices ne, e.

Theorem (C.-Ingalls 2005)

A is a terminal R-order iff it is a full matrix algebra in some $A(n, e)$.

Log terminal orders

From now on $A =$ normal R-order i.e work complete locally.

Theorem (C.-Hacking-Ingalls 2009)

A is log terminal iff $Log(A)$ is log terminal iff A has finite rep type (FRT).

Log terminal max orders classified by Artin in terms of ram data (1987). Log terminal tame orders classified by Reiten-Van den Bergh in terms of AR-quivers (1989).

Proposition(Le Bruyn-Van den Bergh-Van Oystaeyen,1987)

A log terminal order A is reflexive Morita equivalent to $A' = k[[x, y]] *_\eta G$ for some finite $G < GL_2$ & $\eta \in H^2(G, k^*)$. A, A' have same ram data.

- G above is determined by primary ram data. $Z(A) = k[[x, y]]^G$. primary ram data of $A =$ ram data of $k[[x, y]]/k[[x, y]]^G$.
- η is determined by 2ndary ram data.

McKay correspondence for canonical orders

Recall Canonical surface singularities are those of the form $k[[x,y]]^H$ for some finite $H < SL_2$.

Let $A = k[[x,y]] *_\eta G$ be canonical order in skew group ring form as in previous slide.
e.g. $A = k[[x,y]] * H$ is a canonical $k[[x,y]]^H$-order.

Let $f : Z' \longrightarrow \text{Spec } R$ be minimal resolution s.t. $f^\# A$ is terminal.
e.g. above $f : Z' \longrightarrow \text{Spec } k[[x,y]]^H$ is usual min resolution & $f^\# A$ is trivial Azumaya i.e. is $\mathcal{E}nd\ V$ & $\therefore$ Morita equiv to Z'.

Theorem (C. 2010)

The algebras A and $f^\# A$ are derived equivalent.
(except possibly if A has ram type DL)

This gives a corresondence between orbits of reflexive A-modules not containing A & exc curves in the minimal resolution.

Quantum plane curves

Fix $B = A(n, e)$ terminal $k[[u, v]]$-order & $0 \neq f \in k[[u, v]]$.
Study "quantum plane curve" $B/(f)$.

Question

- (FRT) When does $B/(f)$ have finite rep type?
- (AR) If so, what's its AR-quiver?

Answer Matrix factorisation theory tells all. In particular,

Proposition (Knörrer 1987)

Consider double cover $B_f := B[z]/(z^2 - f)$ of B & let $G_f \simeq \mathbb{Z}/2\mathbb{Z}$ be Galois group. Then

$$CM(B_f * G_f)/[B_f] \sim CM(B/(f)).$$

In particular, $B/(f)$ has FRT iff $B_f * G_f$ does.

Quantum plane curves of FRT

Assume $C_f : f = 0$ contains no ram curve of B (else $B/(f)$ not FRT).
Then $B_f * G_f$ is a normal order &

$$\mathrm{Log}(B_f * G_f) = (\mathrm{Spec}\ k[[u,v]], (1 - \frac{1}{ne})C_u + (1 - \frac{1}{e})C_v + \frac{1}{2}C_f).$$

Hence (FRT) question easily reduces to determining ram data of all log terminal $k[[u,v]]$-orders.
Given by easy

Proposition (C.-Ingalls, 201?)

Let A be a log terminal $k[[u,v]]$-order with ram locus C.

- Then C is a simple sing (A,D or E) & $\therefore$ mult $C \leq 3$.
- Possible ram indices classified e.g.
  If $C =$ type A_{2k-1}-node $u^2 = v^{2k}$ with ram indices e_1, e_2, then A is log terminal iff $\{e_1, e_2, k\}$ is a Platonic triple.

Review McKay quivers

Recall for group hom $\rho : G \longrightarrow GL_2$ we have a McKay quiver $Mc(G)$
Vertices = irred representations of G
No. arrows $\rho_1 \to \rho_2 \quad = \dim_k \operatorname{Hom}_G(\rho_1, \rho \otimes \rho_2)$.

More gen, given $\eta \in H^2(G, k^*)$ consider corresponding central extension

$$1 \longrightarrow k^* \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

Can consider the McKay quiver $Mc(G, \eta)$ = full subquiver of $Mc(\tilde{G})$ consisting of those reprn s.t. $k^* < \tilde{G}$ acts by scalar multiplication.

AR-quivers of quantum plane curves

Prop(Knörrer 1987, L-V-V 1987)

The log terminal order $k[[x,y]] *_\eta G$ has AR-quiver $Mc(G,\eta)$.

To find AR-quiver of $B/(f)$ write

$$B \overset{\text{Mor}}{\sim} k[[x,y]] *_\eta G_B, \quad G_B \simeq \mathbb{Z}/ne\mathbb{Z} \times \mathbb{Z}/e\mathbb{Z}$$

One finds easily surj group hom $j : G \longrightarrow G_B \times G_f \longrightarrow G_B$ s.t.

$$B_f * G_f \overset{\text{r.Mori}}{\sim} k[[x,y]] *_{j^*\eta} G.$$

Proposition (C.-Ingalls 201?)

The AR-quiver of $B/(f)$ is the full subquiver of $Mc(G, j^*\eta)$ obtained by deleting the vertices of $Mc(G_B,\eta)$.

i.e. Just remove the AR-quiver of B from $B_f * G_f$.

End

Thank you!

McKay quivers of (G, η)

- $Mc(G, \eta) = \mathbb{Z}\Delta/\text{auto}$ for some ext Dynkin quiver Δ Reiten-Van den Bergh (1989).
- $Mc(G)$ computed for $G < GL_2$ Auslander-Reiten (1986)

We wish to determine all $Mc(G, \eta)$ explicitly for all $G < GL_2$ finite, $\eta \in H^2(G, k^*)$.

Case G **non-abelian** $G = (\mu_{ab} \times_{\mu_a} G_1)/\mu_2$
for some finite $G_1 < SL_2$.

- We have computed $H^2(G, k^*)$ in all cases.
-
$$Mc(G, \eta) = (\mathbb{Z} \times \Delta_H)^{ev} / \langle [+m] \times \phi_1 \phi_2 \rangle$$
for some $H < G_1$ depending on η
$[+m]$ translation on $\mathbb{Z} = Mc(k^*)$
ϕ_1, ϕ_2 automorphisms of $\Delta_H = Mc(H)$. ϕ_1 induced by character of H. ϕ_2 induces by outer automorphism of H

Case G **abelian** $G = \mu_{ab} \times_{\mu_a} \mu_{ac}$
$H^2(G, k^*) \simeq \mu_d, \quad d = \gcd(b, c)$. $Mc(G, \eta) = (\mathbb{Z} \oplus \mathbb{Z})/L$