Non-commutative Mori Contractions

Daniel Chan reporting on
joint work with Adam Nyman

University of New South Wales

April 2009

What is non-commutative projective geometry?

always work over $k = \mathbb{C}$

always work over $k = \mathbb{C}$

Artin-Zhang's viewpoint

What is non-commutative projective geometry?

always work over $k = \mathbb{C}$

Artin-Zhang's viewpoint

Let $A =$ noetherian conn. graded k-algebra, fin gen in deg 1.

What is non-commutative projective geometry?

always work over $k = \mathbb{C}$

Artin-Zhang's viewpoint

Let $A =$ noetherian conn. graded k-algebra, fin gen in deg 1.

Define Proj $A := \operatorname{Gr} A/\text{tors}$

What is non-commutative projective geometry?

always work over $k = \mathbb{C}$

Artin-Zhang's viewpoint

Let $A =$ noetherian conn. graded k-algebra, fin gen in deg 1.

Define $\operatorname{Proj} A := \operatorname{Gr} A/\text{tors}$

where $\operatorname{Gr} A =$ category of graded right A-modules

always work over $k = \mathbb{C}$

Artin-Zhang's viewpoint

Let $A =$ noetherian conn. graded k-algebra, fin gen in deg 1.

Define $\text{Proj}\, A := \text{Gr}\, A/\text{tors}$

where $\text{Gr}\, A =$ category of graded right A-modules

tors $=$ Serre sub-category of $A_{>0}$-torsion modules M

What is non-commutative projective geometry?

always work over $k = \mathbb{C}$

Artin-Zhang's viewpoint

Let $A =$ noetherian conn. graded k-algebra, fin gen in deg 1.

Define $\operatorname{Proj} A := \operatorname{Gr} A/\text{tors}$

where $\operatorname{Gr} A =$ category of graded right A-modules

tors $=$ Serre sub-category of $A_{>0}$-torsion modules M

i.e. each element of M annihilated by some power of $A_{>0}$.

What is non-commutative projective geometry?

always work over $k = \mathbb{C}$

Artin-Zhang's viewpoint

Let $A =$ noetherian conn. graded k-algebra, fin gen in deg 1.

Define Proj $A := \text{Gr}\, A/\text{tors}$

where $\text{Gr}\, A =$ category of graded right A-modules

tors $=$ Serre sub-category of $A_{>0}$-torsion modules M

i.e. each element of M annihilated by some power of $A_{>0}$.

We think of Proj A as a nc proj scheme with homog co-ord ring A since,

always work over $k = \mathbb{C}$

Artin-Zhang's viewpoint

Let $A =$ noetherian conn. graded k-algebra, fin gen in deg 1.

Define Proj $A := \mathrm{Gr}\, A/\mathrm{tors}$

where $\mathrm{Gr}\, A =$ category of graded right A-modules

tors $=$ Serre sub-category of $A_{>0}$-torsion modules M

i.e. each element of M annihilated by some power of $A_{>0}$.

We think of Proj A as a nc proj scheme with homog co-ord ring A since, when A is the homog coord ring of proj scheme Y,

What is non-commutative projective geometry?

always work over $k = \mathbb{C}$

Artin-Zhang's viewpoint

Let $A =$ noetherian conn. graded k-algebra, fin gen in deg 1.

Define Proj $A := \operatorname{Gr} A/\text{tors}$

where $\operatorname{Gr} A =$ category of graded right A-modules

tors $=$ Serre sub-category of $A_{>0}$-torsion modules M

i.e. each element of M annihilated by some power of $A_{>0}$.

We think of Proj A as a nc proj scheme with homog co-ord ring A since, when A is the homog coord ring of proj scheme Y,

Serre's thm $\Longrightarrow$ Proj $A \simeq$ category of quasi-coherent sheaves on Y.

Some examples

The first examples of nc algebras studied from this viewpoint were

Some examples

The first examples of nc algebras studied from this viewpoint were

E.g. 1 3-dim Auslander-regular algebras.

Some examples

The first examples of nc algebras studied from this viewpoint were

E.g. 1 3-dim Auslander-regular algebras. Generic ones are Sklyanin

$$A = Skl(a,b,c) := k\langle x_0, x_1, x_2\rangle / (ax_i x_{i+1} + bx_{i+1}x_i + cx_{i+2}^2)_{i \equiv 0 \mod 3}^2$$

for generic $(a : b : c) \in \mathbb{P}^2$.

The first examples of nc algebras studied from this viewpoint were

E.g. 1 3-dim Auslander-regular algebras. Generic ones are Sklyanin

$$A = Skl(a,b,c) := k\langle x_0, x_1, x_2\rangle / (ax_i x_{i+1} + bx_{i+1}x_i + cx_{i+2}^2)_{i\equiv 0 \mod 3}^2$$

for generic $(a:b:c) \in \mathbb{P}^2$.
This behaves like the polynomial ring in 3 variables since

Some examples

The first examples of nc algebras studied from this viewpoint were

E.g. 1 3-dim Auslander-regular algebras. Generic ones are Sklyanin

$$A = Skl(a,b,c) := k\langle x_0, x_1, x_2\rangle / (ax_i x_{i+1} + bx_{i+1}x_i + cx_{i+2}^2)_{i\equiv 0 \mod 3}^2$$

for generic $(a:b:c) \in \mathbb{P}^2$.
This behaves like the polynomial ring in 3 variables since

1. A has 3 gen & 3 quadratic relations.

Some examples

The first examples of nc algebras studied from this viewpoint were

E.g. 1 3-dim Auslander-regular algebras. Generic ones are Sklyanin

$$A = Skl(a,b,c) := k\langle x_0, x_1, x_2\rangle/(ax_ix_{i+1} + bx_{i+1}x_i + cx_{i+2}^2)_{i\equiv 0 \mod 3}^2$$

for generic $(a:b:c) \in \mathbb{P}^2$.
This behaves like the polynomial ring in 3 variables since

1. A has 3 gen & 3 quadratic relations.
2. A is noetherian.

Some examples

The first examples of nc algebras studied from this viewpoint were

E.g. 1 3-dim Auslander-regular algebras. Generic ones are Sklyanin

$$A = Skl(a, b, c) := k\langle x_0, x_1, x_2\rangle/(ax_i x_{i+1} + bx_{i+1}x_i + cx_{i+2}^2)_{i\equiv 0 \mod 3}^2$$

for generic $(a : b : c) \in \mathbb{P}^2$.
This behaves like the polynomial ring in 3 variables since

1. A has 3 gen & 3 quadratic relations.
2. A is noetherian.
3. $\dim_k A_n = \binom{n+2}{2} = \dim_k k[x_0, x_1, x_2]_n$.

Some examples

The first examples of nc algebras studied from this viewpoint were

E.g. 1 3-dim Auslander-regular algebras. Generic ones are Sklyanin

$$A = Skl(a,b,c) := k\langle x_0, x_1, x_2\rangle/(ax_ix_{i+1} + bx_{i+1}x_i + cx_{i+2}^2)_{i\equiv 0 \mod 3}^2$$

for generic $(a:b:c) \in \mathbb{P}^2$.
This behaves like the polynomial ring in 3 variables since

1. A has 3 gen & 3 quadratic relations.
2. A is noetherian.
3. $\dim_k A_n = \binom{n+2}{2} = \dim_k k[x_0, x_1, x_2]_n$.
4. A has global dimension 3.

The first examples of nc algebras studied from this viewpoint were

E.g. 1 3-dim Auslander-regular algebras. Generic ones are Sklyanin

$$A = Skl(a, b, c) := k\langle x_0, x_1, x_2\rangle/(ax_ix_{i+1} + bx_{i+1}x_i + cx_{i+2}^2)_{i\equiv 0 \mod 3}^2$$

for generic $(a : b : c) \in \mathbb{P}^2$.
This behaves like the polynomial ring in 3 variables since

1. A has 3 gen & 3 quadratic relations.
2. A is noetherian.
3. $\dim_k A_n = \binom{n+2}{2} = \dim_k k[x_0, x_1, x_2]_n$.
4. A has global dimension 3.

We think of Proj A in this case as a nc $\mathbb{P}^2$.

E.g.2 There are also 4-dim Sklyanin algebras Skl_4

E.g.2 There are also 4-dim Sklyanin algebras Skl_4 such as homogenised $U(\mathfrak{sl}_2)$

$$A = \frac{k\langle e, f, h, z\rangle}{(ef - fe - zh, eh - he - 2ze, fh - hf + 2zf, ze - ez, zh - hz, zf - fz)}.$$

E.g.2 There are also 4-dim Sklyanin algebras Skl_4 such as homogenised $U(\mathfrak{sl}_2)$

$$A = \frac{k\langle e, f, h, z\rangle}{(ef - fe - zh, eh - he - 2ze, fh - hf + 2zf, ze - ez, zh - hz, zf - fz)}.$$

We think of Proj Skl_4 as a nc $\mathbb{P}^3$.

E.g.2 There are also 4-dim Sklyanin algebras Skl_4 such as homogenised $U(\mathfrak{sl}_2)$

$$A = \frac{k\langle e, f, h, z\rangle}{(ef - fe - zh, eh - he - 2ze, fh - hf + 2zf, ze - ez, zh - hz, zf - fz)}.$$

We think of Proj Skl_4 as a nc $\mathbb{P}^3$.

Generically, $\dim_k Z(A)_2 = 2$. Given $t \in Z(A)_2$,

E.g.2 There are also 4-dim Sklyanin algebras Skl_4 such as homogenised $U(\mathfrak{sl}_2)$

$$A = \frac{k\langle e,f,h,z\rangle}{(ef-fe-zh, eh-he-2ze, fh-hf+2zf, ze-ez, zh-hz, zf-fz)}.$$

We think of Proj Skl_4 as a nc $\mathbb{P}^3$.

Generically, $\dim_k Z(A)_2 = 2$. Given $t \in Z(A)_2$, Proj $A/(t)$ is a nc $\mathbb{P}^1 \times \mathbb{P}^1$.

Non-commutative projective curves

Given a graded vector space M, we can define its *Gelfand-Kirillov dimension*

Given a graded vector space M, we can define its *Gelfand-Kirillov dimension*

$gk(M) = \text{growth rate of } f_M(n) := \dim_k(\oplus_{i\leq n} M_i)$

Given a graded vector space M, we can define its *Gelfand-Kirillov dimension*

$$gk(M) = \text{growth rate of } f_M(n) := \dim_k(\oplus_{i\leq n} M_i)$$
$$= \inf\{c | f_M(n) \leq n^c, n \gg 0\}.$$

Given a graded vector space M, we can define its *Gelfand-Kirillov dimension*

$$gk(M) = \text{growth rate of } f_M(n) := \dim_k(\oplus_{i \le n} M_i) \\ = \inf\{c | f_M(n) \le n^c, n \gg 0\}.$$

Classification of non-commutative proj curves is essentially answered by following

Given a graded vector space M, we can define its *Gelfand-Kirillov dimension*

$$gk(M) = \text{growth rate of } f_M(n) := \dim_k(\oplus_{i \leq n} M_i)$$
$$= \inf\{c | f_M(n) \leq n^c, n \gg 0\}.$$

Classification of non-commutative proj curves is essentially answered by following

Theorem (Artin-Stafford 1995)

Let A be a noetherian connected graded k-domain of Gelfand-Kirillov dimension 2.

Non-commutative projective curves

Given a graded vector space M, we can define its *Gelfand-Kirillov dimension*

$$gk(M) = \text{growth rate of } f_M(n) := \dim_k(\oplus_{i\leq n} M_i)$$
$$= \inf\{c | f_M(n) \leq n^c, n \gg 0\}.$$

Classification of non-commutative proj curves is essentially answered by following

Theorem (Artin-Stafford 1995)

Let A be a noetherian connected graded k-domain of Gelfand-Kirillov dimension 2. Then there is a projective curve Y and a category equivalence,

$$\text{Proj } A \simeq \text{QCoh}(Y).$$

Let A be a noetherian connected graded k-domain with gk-dimension three satisfying some nice homological properties (see later),

Artin's conjecture on birational classification of nc surfaces

Let A be a noetherian connected graded k-domain with gk-dimension three satisfying some nice homological properties (see later),

$Q(A) =$ graded ring of fractions.

Let A be a noetherian connected graded k-domain with gk-dimension three satisfying some nice homological properties (see later),

$Q(A) =$ graded ring of fractions.

Then Artin's conjecture (1997) can be loosely re-stated as

Let A be a noetherian connected graded k-domain with gk-dimension three satisfying some nice homological properties (see later),

$Q(A) =$ graded ring of fractions.

Then Artin's conjecture (1997) can be loosely re-stated as

Conjecture

The division ring $Q(A)_0$ falls into one of the following categories.

Let A be a noetherian connected graded k-domain with gk-dimension three satisfying some nice homological properties (see later),

$Q(A) =$ graded ring of fractions.

Then Artin's conjecture (1997) can be loosely re-stated as

Conjecture

The division ring $Q(A)_0$ falls into one of the following categories.

1. $Q(A)_0 = Q(S)_0$ where S is 3-dim Auslander-regular i.e. Proj A is "birationally del Pezzo".

Artin's conjecture on birational classification of nc surfaces

Let A be a noetherian connected graded k-domain with gk-dimension three satisfying some nice homological properties (see later),

$Q(A) =$ graded ring of fractions.

Then Artin's conjecture (1997) can be loosely re-stated as

Conjecture

The division ring $Q(A)_0$ falls into one of the following categories.

1. $Q(A)_0 = Q(S)_0$ where S is 3-dim Auslander-regular i.e. Proj A is "birationally del Pezzo".
2. There's a curve E and $Q(A)_0$ is the field of fractions of either the Öre extension $k(E)[t;\sigma]$ or the ring of differential operators $D(E)$ i.e. Proj A is "birationally ruled".

Let A be a noetherian connected graded k-domain with gk-dimension three satisfying some nice homological properties (see later),

$Q(A) =$ graded ring of fractions.

Then Artin's conjecture (1997) can be loosely re-stated as

Conjecture

The division ring $Q(A)_0$ falls into one of the following categories.

1. $Q(A)_0 = Q(S)_0$ where S is 3-dim Auslander-regular i.e. Proj A is "birationally del Pezzo".
2. There's a curve E and $Q(A)_0$ is the field of fractions of either the Öre extension $k(E)[t;\sigma]$ or the ring of differential operators $D(E)$ i.e. Proj A is "birationally ruled".
3. $Q(A)_0$ is finite over its centre.

Aim of Talk

Cases 1) and 2) of the conjecture are reminiscent of the possibilities that can occur

Aim of Talk

Cases 1) and 2) of the conjecture are reminiscent of the possibilities that can occur for the extremal contraction of a K-negative curve C on a surface Y where

Aim of Talk

Cases 1) and 2) of the conjecture are reminiscent of the possibilities that can occur for the extremal contraction of a K-negative curve C on a surface Y where 1) $C^2 > 0$ or 2) $C^2 = 0$.

Aim of Talk

Cases 1) and 2) of the conjecture are reminiscent of the possibilities that can occur for the extremal contraction of a K-negative curve C on a surface Y where 1) $C^2 > 0$ or 2) $C^2 = 0$.

Aim

We wish to establish a contraction theorem on nc surfaces, analogous to case 2).

Aim of Talk

Cases 1) and 2) of the conjecture are reminiscent of the possibilities that can occur for the extremal contraction of a K-negative curve C on a surface Y where 1) $C^2 > 0$ or 2) $C^2 = 0$.

Aim

We wish to establish a contraction theorem on nc surfaces, analogous to case 2).

We need to define nc smooth projective surfaces, dualising sheaves, intersection numbers etc.

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.
$Y =$ Proj A is a *nc smooth proj d-fold* if further

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.
$Y =$ Proj A is a *nc smooth proj d-fold* if further

1. A is a domain.

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.
$Y =$ Proj A is a *nc smooth proj d-fold* if further

1. A is a domain.
2. A is strongly noetherian i.e. $A \otimes_k R$ noeth for any comm noeth k-algebra R.

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.
$Y =$ Proj A is a *nc smooth proj d-fold* if further

1. A is a domain.
2. A is strongly noetherian i.e. $A \otimes_k R$ noeth for any comm noeth k-algebra R.
3. (Gorenstein) there are A-bimodules ω_Y, ω_Y^{-1} s.t. $\omega_Y[d+1]$ is an Auslander balanced dualising complex &

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.
$Y =$ Proj A is a *nc smooth proj d-fold* if further

1. A is a domain.
2. A is strongly noetherian i.e. $A \otimes_k R$ noeth for any comm noeth k-algebra R.
3. (Gorenstein) there are A-bimodules ω_Y, ω_Y^{-1} s.t. $\omega_Y[d+1]$ is an Auslander balanced dualising complex & $- \otimes_A \omega_Y, - \otimes_A \omega_Y^{-1}$ induce inverse auto-equivalences on Proj A (more later).

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.
$Y =$ Proj A is a *nc smooth proj d-fold* if further

1. A is a domain.
2. A is strongly noetherian i.e. $A \otimes_k R$ noeth for any comm noeth k-algebra R.
3. (Gorenstein) there are A-bimodules ω_Y, ω_Y^{-1} s.t. $\omega_Y[d+1]$ is an Auslander balanced dualising complex & $- \otimes_A \omega_Y, - \otimes_A \omega_Y^{-1}$ induce inverse auto-equivalences on Proj A (more later).
4. (Smooth) $\mathrm{Ext}_Y^{d+1}(-,-) = 0$. N.B. Mod Y has enough injectives.

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.
$Y =$ Proj A is a *nc smooth proj d-fold* if further

1. A is a domain.
2. A is strongly noetherian i.e. $A \otimes_k R$ noeth for any comm noeth k-algebra R.
3. (Gorenstein) there are A-bimodules ω_Y, ω_Y^{-1} s.t. $\omega_Y[d+1]$ is an Auslander balanced dualising complex & $- \otimes_A \omega_Y, - \otimes_A \omega_Y^{-1}$ induce inverse auto-equivalences on Proj A (more later).
4. (Smooth) $\mathrm{Ext}_Y^{d+1}(-,-) = 0$. N.B. Mod Y has enough injectives.
5. (Macaulay) The Gelfand-Kirillov and canonical dimension coincide (more later).

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.
$Y =$ Proj A is a *nc smooth proj d-fold* if further

1. A is a domain.
2. A is strongly noetherian i.e. $A \otimes_k R$ noeth for any comm noeth k-algebra R.
3. (Gorenstein) there are A-bimodules ω_Y, ω_Y^{-1} s.t. $\omega_Y[d+1]$ is an Auslander balanced dualising complex & $- \otimes_A \omega_Y, - \otimes_A \omega_Y^{-1}$ induce inverse auto-equivalences on Proj A (more later).
4. (Smooth) $\mathrm{Ext}_Y^{d+1}(-,-) = 0$. N.B. Mod Y has enough injectives.
5. (Macaulay) The Gelfand-Kirillov and canonical dimension coincide (more later).

E.g. Proj Skl_4 is a nc smooth projective 3-fold.
For generic $t \in Z(Skl_4)_2$,

Technical assumptions on A

Let $A =$ conn graded k-algebra, fin gen in degree 1.
$Y =$ Proj A is a *nc smooth proj d-fold* if further

1. A is a domain.
2. A is strongly noetherian i.e. $A \otimes_k R$ noeth for any comm noeth k-algebra R.
3. (Gorenstein) there are A-bimodules ω_Y, ω_Y^{-1} s.t. $\omega_Y[d+1]$ is an Auslander balanced dualising complex & $- \otimes_A \omega_Y, - \otimes_A \omega_Y^{-1}$ induce inverse auto-equivalences on Proj A (more later).
4. (Smooth) $\mathrm{Ext}_Y^{d+1}(-,-) = 0$. N.B. Mod Y has enough injectives.
5. (Macaulay) The Gelfand-Kirillov and canonical dimension coincide (more later).

E.g. Proj Skl_4 is a nc smooth projective 3-fold.
For generic $t \in Z(Skl_4)_2$, Proj $Skl_4/(t)$ and Proj $Skl(a,b,c)$ are nc smooth projective surfaces.

Geometric techniques available

Let Y = nc smooth proj surface,

Geometric techniques available

Let Y = nc smooth proj surface,

Mod Y = category Y, mod Y = full subcat of noeth objects.

Geometric techniques available

Let Y = nc smooth proj surface,

Mod Y = category Y, mod Y = full subcat of noeth objects.

Then we have the following geometric concepts for Y

1. cohomology

Geometric techniques available

Let Y = nc smooth proj surface,

Mod Y = category Y, mod Y = full subcat of noeth objects.

Then we have the following geometric concepts for Y

1. cohomology
2. dualising complex and Serre duality

Geometric techniques available

Let Y = nc smooth proj surface,

Mod Y = category Y, mod Y = full subcat of noeth objects.

Then we have the following geometric concepts for Y

1. cohomology
2. dualising complex and Serre duality
3. dimension function dim M, $M \in$ mod Y

Geometric techniques available

Let Y = nc smooth proj surface,

Mod Y = category Y, mod Y = full subcat of noeth objects.

Then we have the following geometric concepts for Y

1. cohomology
2. dualising complex and Serre duality
3. dimension function $\dim M$, $M \in \text{mod } Y$
4. intersection theory

Geometric techniques available

Let Y = nc smooth proj surface,

Mod Y = category Y, mod Y = full subcat of noeth objects.

Then we have the following geometric concepts for Y

1. cohomology
2. dualising complex and Serre duality
3. dimension function dim M, $M \in$ mod Y
4. intersection theory
5. Hilbert schemes

Cohomology (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold,

Let $Y = \text{Proj } A =$ nc smooth proj d-fold,

$$A =: \mathcal{O}_{\mathsf{Y}} \in \text{mod } Y.$$

Cohomology (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold,

$$A =: \mathcal{O}_Y \in \text{mod } Y.$$

For $M \in \text{Mod } Y$,

$$H^i(M) := \text{Ext}^i_Y(\mathcal{O}_Y, M).$$

Let $Y = \text{Proj } A =$ nc smooth proj d-fold,

$$A =: \mathcal{O}_{\mathrm{Y}} \in \text{mod } Y.$$

For $M \in \text{Mod } Y$,

$$H^i(M) := \text{Ext}^i_{\mathrm{Y}}(\mathcal{O}_{\mathrm{Y}}, M).$$

Thm (Artin-Zhang 94)

For $M, N \in \text{mod } Y$, $H^i(M)$

Cohomology (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold,

$$A =: \mathcal{O}_Y \in \text{mod } Y.$$

For $M \in \text{Mod } Y$,

$$H^i(M) := \text{Ext}^i_Y(\mathcal{O}_Y, M).$$

Thm (Artin-Zhang 94)

For $M, N \in \text{mod } Y$, $H^i(M)$ and more generally $\text{Ext}^i_Y(N, M)$ are finite dimensional.

Dualising complex and Serre duality

Studied by Yekutieli, Zhang, Van den Bergh etc. since 1990.

Dualising complex and Serre duality

Studied by Yekutieli, Zhang, Van den Bergh etc. since 1990.

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.

Dualising complex and Serre duality

Studied by Yekutieli, Zhang, Van den Bergh etc. since 1990.

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Define $A_{>0}$-torsion functor $\Gamma_{\mathfrak{m}} : \operatorname{Gr} A \longrightarrow \operatorname{tors}$ by

$$\Gamma_{\mathfrak{m}} = \lim_{n \to \infty} \operatorname{Hom}_A(A/A_{\geq n}, -).$$

Dualising complex and Serre duality

Studied by Yekutieli, Zhang, Van den Bergh etc. since 1990.

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Define $A_{>0}$-torsion functor $\Gamma_{\mathfrak{m}} : \operatorname{Gr} A \longrightarrow \operatorname{tors}$ by

$$\Gamma_{\mathfrak{m}} = \lim_{n \to \infty} \operatorname{Hom}_A(A/A_{\geq n}, -).$$

Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^{\vee} \in D^{+}(A^o \otimes_k A)$.

Dualising complex and Serre duality

Studied by Yekutieli, Zhang, Van den Bergh etc. since 1990.

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Define $A_{>0}$-torsion functor $\Gamma_{\mathfrak{m}} : \mathrm{Gr}\, A \longrightarrow$ tors by

$$\Gamma_{\mathfrak{m}} = \lim_{n \to \infty} \mathrm{Hom}_A(A/A_{\geq n}, -).$$

Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^{\vee} \in D^+(A^o \otimes_k A)$.
It is well-behaved under certain technical conditions (omitted), in which case we will say that ω_A is an *Auslander balanced dualising complex*.

Dualising complex and Serre duality

Studied by Yekutieli, Zhang, Van den Bergh etc. since 1990.

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Define $A_{>0}$-torsion functor $\Gamma_{\mathfrak{m}} : \operatorname{Gr} A \longrightarrow \text{tors}$ by

$$\Gamma_{\mathfrak{m}} = \lim_{n\to\infty} \operatorname{Hom}_A(A/A_{\geq n}, -).$$

Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^\vee \in D^+(A^o \otimes_k A)$.
It is well-behaved under certain technical conditions (omitted), in which case we will say that ω_A is an *Auslander balanced dualising complex*.

If $Y = \operatorname{Proj} A =$ nc smooth proj d-fold,

then there are the usual (Bondal-Kapranov-)Serre duality isomorphisms

Dualising complex and Serre duality

Studied by Yekutieli, Zhang, Van den Bergh etc. since 1990.

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Define $A_{>0}$-torsion functor $\Gamma_{\mathfrak{m}} : \operatorname{Gr} A \longrightarrow \operatorname{tors}$ by

$$\Gamma_{\mathfrak{m}} = \lim_{n\to\infty} \operatorname{Hom}_A(A/A_{\geq n}, -).$$

Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^{\vee} \in D^{+}(A^{o} \otimes_k A)$.
It is well-behaved under certain technical conditions (omitted), in which case we will say that ω_A is an *Auslander balanced dualising complex*.

If $Y = \operatorname{Proj} A =$ nc smooth proj d-fold,

then there are the usual (Bondal-Kapranov-)Serre duality isomorphisms

$$\operatorname{Ext}^i_Y(M, N) \simeq \operatorname{Ext}^{d-i}_Y(N, M \otimes \omega_Y)^*$$

natural in $M, N \in \operatorname{mod} Y, \omega_A = \omega_Y[d+1]$.

Dimension of sheaves

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.

Dimension of sheaves

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.

Canonical dimension: For noetherian $M \in \operatorname{Gr} A$,

Dimension of sheaves

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.

Canonical dimension: For noetherian $M \in \operatorname{Gr} A$,

$$c.\dim(M) := \max\{i | R^i\Gamma_{\mathfrak{m}}(M) \neq 0\}.$$

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.

Canonical dimension: For noetherian $M \in \operatorname{Gr} A$,

$$c.\dim(M) := \max\{i | R^i\Gamma_{\mathfrak{m}}(M) \neq 0\}.$$

On $Y = \operatorname{Proj} A$, we can sensibly define

$$\dim = gk - 1 \quad \text{or} \quad c.\dim - 1$$

Intersection theory (I.Mori-Smith)

Let $Y = \text{Proj } A =$ nc smooth proj surface.

Intersection theory (I.Mori-Smith)

Let $Y = \text{Proj } A =$ nc smooth proj surface.

For $M, N \in \text{mod } Y$, we have the well-defined intersection pairing

$$M.N := -\sum_{i=0}^{2} (-1)^i \dim \text{Ext}^i_Y(M, N)$$

Intersection theory (I.Mori-Smith)

Let $Y = \text{Proj } A =$ nc smooth proj surface.

For $M, N \in \text{mod } Y$, we have the well-defined intersection pairing

$$M.N := -\sum_{i=0}^{2}(-1)^i \dim \text{Ext}^i_Y(M, N)$$

Motivation If Y is comm, & $C, D \subset Y$ are curves

Let $Y = \text{Proj } A =$ nc smooth proj surface.

For $M, N \in \text{mod } Y$, we have the well-defined intersection pairing

$$M.N := -\sum_{i=0}^{2} (-1)^i \dim \text{Ext}^i_Y(M, N)$$

Motivation If Y is comm, & $C, D \subset Y$ are curves

then $\mathcal{O}_C \, . \, \mathcal{O}_D = C.D$.

Hilbert schemes (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

Hilbert schemes (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

For comm k-algebra R can define

$$\text{Mod } Y_R = \text{Proj } A_R = \text{Gr}(A \otimes_k R)/\text{tors}$$

Hilbert schemes (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

For comm k-algebra R can define

$$\text{Mod } Y_R = \text{Proj } A_R = \text{Gr}(A \otimes_k R)/\text{tors}$$

For $\mathcal{M} \in \text{Mod } Y_R$ can define $\mathcal{M} \otimes_R - : \text{Mod } R \longrightarrow \text{Mod } Y_R$

Hilbert schemes (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

For comm k-algebra R can define

$$\text{Mod } Y_R = \text{Proj } A_R = \text{Gr}(A \otimes_k R)/\text{tors}$$

For $\mathcal{M} \in \text{Mod } Y_R$ can define $\mathcal{M} \otimes_R - : \text{Mod } R \longrightarrow \text{Mod } Y_R$
& thus flatness $/R$ as well as the Hilbert functor of flat quotients.

Hilbert schemes (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

For comm k-algebra R can define

$$\text{Mod } Y_R = \text{Proj } A_R = \text{Gr}(A \otimes_k R)/\text{tors}$$

For $\mathcal{M} \in \text{Mod } Y_R$ can define $\mathcal{M} \otimes_R - : \text{Mod } R \longrightarrow \text{Mod } Y_R$
& thus flatness $/R$ as well as the Hilbert functor of flat quotients.

Theorem (loosely stated Artin-Zhang 2001)

For $P \in \text{mod } Y$, there exists a Hilbert scheme $\text{Hilb } P$ parametrising quotients of P.

Hilbert schemes (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

For comm k-algebra R can define

$$\text{Mod } Y_R = \text{Proj } A_R = \text{Gr}(A \otimes_k R)/\text{tors}$$

For $\mathcal{M} \in \text{Mod } Y_R$ can define $\mathcal{M} \otimes_R - : \text{Mod } R \longrightarrow \text{Mod } Y_R$
& thus flatness $/R$ as well as the Hilbert functor of flat quotients.

Theorem (loosely stated Artin-Zhang 2001)

For $P \in \text{mod } Y$, there exists a Hilbert scheme $\text{Hilb } P$ parametrising quotients of P. $\text{Hilb } P$ is a countable union of projective schemes which is locally of finite type.

K-non-effective rational curves with self-intersection 0

Let $Y =$ nc smooth proj surface.

K-non-effective rational curves with self-intersection 0

Let Y = nc smooth proj surface.

Say $M \in \text{mod}\, Y$ is a *K-non-effective rational curve with self-intersection 0* if

K-non-effective rational curves with self-intersection 0

Let Y = nc smooth proj surface.

Say $M \in \text{mod}\, Y$ is a *K-non-effective rational curve with self-intersection 0* if

1. M is a 1-critical quotient of $\mathcal{O}_Y$.

K-non-effective rational curves with self-intersection 0

Let Y = nc smooth proj surface.

Say $M \in \text{mod}\, Y$ is a *K-non-effective rational curve with self-intersection 0* if

1. M is a 1-critical quotient of $\mathcal{O}_Y$.
2. $H^0(M) = k, \; H^1(M) = 0, \; M^2 = 0.$

K-non-effective rational curves with self-intersection 0

Let Y = nc smooth proj surface.

Say $M \in \text{mod}\, Y$ is a *K-non-effective rational curve with self-intersection 0* if

1. M is a 1-critical quotient of $\mathcal{O}_Y$.
2. $H^0(M) = k, \; H^1(M) = 0, \; M^2 = 0$.
3. $H^0(M \otimes \omega_Y) = 0$.

K-non-effective rational curves with self-intersection 0

Let $Y =$ nc smooth proj surface.

Say $M \in \text{mod } Y$ is a *K-non-effective rational curve with self-intersection 0* if

1. M is a 1-critical quotient of $\mathcal{O}_Y$.
2. $H^0(M) = k, \; H^1(M) = 0, \; M^2 = 0$.
3. $H^0(M \otimes \omega_Y) = 0$.

N.B. In nc case, we don't know if $K.C < 0 \implies H^0(\mathcal{O}_C \otimes \omega) = 0$.

K-non-effective rational curves with self-intersection 0

Let $Y =$ nc smooth proj surface.

Say $M \in \text{mod}\, Y$ is a *K-non-effective rational curve with self-intersection 0* if

1. M is a 1-critical quotient of $\mathcal{O}_{\mathsf{Y}}$.
2. $H^0(M) = k,\ H^1(M) = 0,\ M^2 = 0.$
3. $H^0(M \otimes \omega_Y) = 0.$

N.B. In nc case, we don't know if $K.C < 0 \Longrightarrow H^0(\mathcal{O}_C \otimes \omega) = 0.$

Theorem (C-Nyman)

The point p of $\text{Hilb}\, \mathcal{O}_{\mathsf{Y}}$ corresponding to M is smooth and has 1-dim tangent space.

K-non-effective rational curves with self-intersection 0

Let $Y =$ nc smooth proj surface.

Say $M \in \operatorname{mod} Y$ is a *K-non-effective rational curve with self-intersection 0* if

1. M is a 1-critical quotient of $\mathcal{O}_Y$.
2. $H^0(M) = k, \; H^1(M) = 0, \; M^2 = 0.$
3. $H^0(M \otimes \omega_Y) = 0.$

N.B. In nc case, we don't know if $K.C < 0 \Longrightarrow H^0(\mathcal{O}_C \otimes \omega) = 0.$

Theorem (C-Nyman)

The point p of $\operatorname{Hilb} \mathcal{O}_Y$ corresponding to M is smooth and has 1-dim tangent space.

Let X be the component of $\operatorname{Hilb} \mathcal{O}_Y$ containing the point p in the thm.

K-non-effective rational curves with self-intersection 0

Let Y = nc smooth proj surface.

Say $M \in \text{mod } Y$ is a *K-non-effective rational curve with self-intersection 0* if

1. M is a 1-critical quotient of $\mathcal{O}_Y$.
2. $H^0(M) = k, \ H^1(M) = 0, \ M^2 = 0.$
3. $H^0(M \otimes \omega_Y) = 0.$

N.B. In nc case, we don't know if $K.C < 0 \Longrightarrow H^0(\mathcal{O}_C \otimes \omega) = 0$.

Theorem (C-Nyman)

The point p of $\text{Hilb}\,\mathcal{O}_Y$ corresponding to M is smooth and has 1-dim tangent space.

Let X be the component of $\text{Hilb}\,\mathcal{O}_Y$ containing the point p in the thm. We wish to construct a morphism $f : Y \longrightarrow X$ i.e.

K-non-effective rational curves with self-intersection 0

Let Y = nc smooth proj surface.

Say $M \in \text{mod}\, Y$ is a *K-non-effective rational curve with self-intersection 0* if

1. M is a 1-critical quotient of $\mathcal{O}_Y$.
2. $H^0(M) = k,\ H^1(M) = 0,\ M^2 = 0.$
3. $H^0(M \otimes \omega_Y) = 0.$

N.B. In nc case, we don't know if $K.C < 0 \Longrightarrow H^0(\mathcal{O}_C \otimes \omega) = 0.$

Theorem (C-Nyman)

The point p of $\text{Hilb}\, \mathcal{O}_Y$ corresponding to M is smooth and has 1-dim tangent space.

Let X be the component of $\text{Hilb}\, \mathcal{O}_Y$ containing the point p in the thm. We wish to construct a morphism $f : Y \longrightarrow X$ i.e. adjoint functors

$$f^* : \text{Mod}\, X \longrightarrow \text{Mod}\, Y, f_* : \text{Mod}\, Y \longrightarrow \text{Mod}\, X.$$

Digression on $\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

Digression on π_* : Mod $Y_X \longrightarrow$ Mod Y

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

Define π_* via relative Cech cohomology as usual i.e.

Digression on $\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

Define π_* via relative Cech cohomology as usual i.e.

For $\mathcal{N} \in \text{Mod } Y_X$ & open affine cover $X = U \cup V$,

Digression on $\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

Define π_* via relative Cech cohomology as usual i.e.

For $\mathcal{N} \in \text{Mod } Y_X$ & open affine cover $X = U \cup V$,

restriction maps give morphism in Mod Y

Digression on $\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$

Let Y = nc smooth proj d-fold,

X =(integral) proj curve,

Define π_* via relative Cech cohomology as usual i.e.

For $\mathcal{N} \in \text{Mod } Y_X$ & open affine cover $X = U \cup V$,

restriction maps give morphism in Mod Y

$$\mathcal{N}(U) \oplus \mathcal{N}(V) \xrightarrow{d} \mathcal{N}(U \cap V)$$

Digression on $\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

Define π_* via relative Cech cohomology as usual i.e.

For $\mathcal{N} \in \text{Mod } Y_X$ & open affine cover $X = U \cup V$,

restriction maps give morphism in Mod Y

$$\mathcal{N}(U) \oplus \mathcal{N}(V) \xrightarrow{d} \mathcal{N}(U \cap V)$$

Define

$$\pi_*(\mathcal{N}) = \ker d$$

Digression on $\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

Define π_* via relative Cech cohomology as usual i.e.

For $\mathcal{N} \in \text{Mod } Y_X$ & open affine cover $X = U \cup V$,

restriction maps give morphism in Mod Y

$$\mathcal{N}(U) \oplus \mathcal{N}(V) \xrightarrow{d} \mathcal{N}(U \cap V)$$

Define

$$\pi_*(\mathcal{N}) = \ker d$$

N.B. π_* independent of choice of open affine cover.

Digression on $\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

Define π_* via relative Cech cohomology as usual i.e.

For $\mathcal{N} \in \text{Mod } Y_X$ & open affine cover $X = U \cup V$,

restriction maps give morphism in Mod Y

$$\mathcal{N}(U) \oplus \mathcal{N}(V) \xrightarrow{d} \mathcal{N}(U \cap V)$$

Define

$$\pi_*(\mathcal{N}) = \ker d$$

N.B. π_* independent of choice of open affine cover.

For $\mathcal{M} \in \text{mod } Y_X$ flat over X, we wish to consider the "Fourier-Mukai transform"

Digression on $\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$

Let Y = nc smooth proj d-fold,

X =(integral) proj curve,

Define π_* via relative Cech cohomology as usual i.e.

For $\mathcal{N} \in \text{Mod } Y_X$ & open affine cover $X = U \cup V$,

restriction maps give morphism in Mod Y

$$\mathcal{N}(U) \oplus \mathcal{N}(V) \xrightarrow{d} \mathcal{N}(U \cap V)$$

Define

$$\pi_*(\mathcal{N}) = \ker d$$

N.B. π_* independent of choice of open affine cover.

For $\mathcal{M} \in \text{mod } Y_X$ flat over X, we wish to consider the "Fourier-Mukai transform"

$$\text{Mod } X \xrightarrow{\mathcal{M} \otimes_X -} \text{Mod } Y_X \xrightarrow{\pi_*} \text{Mod } Y.$$

Fourier-Mukai morphisms to a curve

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

Fourier-Mukai morphisms to a curve

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

$\mathcal{M}/X =$ flat family of objects in mod Y.

Let Y = nc smooth proj d-fold,

X =(integral) proj curve,

$\mathcal{M}/X$ = flat family of objects in mod Y.

Defn Say $\mathcal{M}$ is *base point free* if for any simple $P \in \text{mod } Y$,

Fourier-Mukai morphisms to a curve

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

$\mathcal{M}/X =$ flat family of objects in mod Y.

Defn Say $\mathcal{M}$ is *base point free* if for any simple $P \in \text{mod}\, Y$, $\text{Hom}_Y(M, P) = 0$ for a generic fibre $M \in \mathcal{M}$.

Let Y = nc smooth proj d-fold,

X =(integral) proj curve,

$\mathcal{M}/X$ = flat family of objects in mod Y.

Defn Say $\mathcal{M}$ is *base point free* if for any simple $P \in \text{mod } Y$, $\text{Hom}_Y(M, P) = 0$ for a generic fibre $M \in \mathcal{M}$.

Theorem (C-Nyman)

If $\mathcal{M}/X$ is base point free then

Fourier-Mukai morphisms to a curve

Let $Y =$ nc smooth proj d-fold,

$X =$(integral) proj curve,

$\mathcal{M}/X =$ flat family of objects in mod Y.

Defn Say $\mathcal{M}$ is *base point free* if for any simple $P \in \text{mod } Y$, $\text{Hom}_Y(M, P) = 0$ for a generic fibre $M \in \mathcal{M}$.

Theorem (C-Nyman)

If $\mathcal{M}/X$ is base point free then

- $f^* = \pi_*(\mathcal{M} \otimes_X -)$ *is exact so has a right adjoint.*

Fourier-Mukai morphisms to a curve

Let Y = nc smooth proj d-fold,

X =(integral) proj curve,

$\mathcal{M}/X$ = flat family of objects in mod Y.

Defn Say $\mathcal{M}$ is *base point free* if for any simple $P \in$ mod Y, $\mathrm{Hom}_Y(M, P) = 0$ for a generic fibre $M \in \mathcal{M}$.

Theorem (C-Nyman)

If $\mathcal{M}/X$ is base point free then

- $f^* = \pi_*(\mathcal{M} \otimes_X -)$ *is exact so has a right adjoint.*
- f^* *preserves noeth objects.*

Fourier-Mukai morphisms to a curve

Let Y = nc smooth proj d-fold,

X =(integral) proj curve,

$\mathcal{M}/X$ = flat family of objects in mod Y.

Defn Say $\mathcal{M}$ is *base point free* if for any simple $P \in \text{mod } Y$, $\text{Hom}_Y(M, P) = 0$ for a generic fibre $M \in \mathcal{M}$.

Theorem (C-Nyman)

If $\mathcal{M}/X$ is base point free then

- $f^* = \pi_*(\mathcal{M} \otimes_X -)$ *is exact so has a right adjoint.*
- f^* *preserves noeth objects.*

The morphism $f : Y \longrightarrow X$ in the thm is called a Fourier-Mukai morphism.

Non-commutative Mori contraction

Let Y = nc smooth proj surface,

Non-commutative Mori contraction

Let $Y =$ nc smooth proj surface,
$M = K$-non-effective rational curve with $M^2 = 0$,

Non-commutative Mori contraction

Let Y = nc smooth proj surface,
$M = K$-non-effective rational curve with $M^2 = 0$,
$\mathcal{M}/X$ comp of $\mathrm{Hilb}\,\mathcal{O}_Y$ containing M.

Non-commutative Mori contraction

Let Y = nc smooth proj surface,
$M = K$-non-effective rational curve with $M^2 = 0$,
$\mathcal{M}/X$ comp of $\mathrm{Hilb}\,\mathcal{O}_Y$ containing M.

Theorem (MAIN, C-Nyman)

If for every simple 0-dim quotient $P \in \mathrm{mod}\,Y$ of M we have $M.P = 0$,

Non-commutative Mori contraction

Let $Y =$ nc smooth proj surface,
$M = K$-non-effective rational curve with $M^2 = 0$,
$\mathcal{M} / X$ comp of Hilb $\mathcal{O}_Y$ containing M.

Theorem (MAIN, C-Nyman)

If for every simple 0-dim quotient $P \in \mathrm{mod}\, Y$ of M we have $M.P = 0$, then $\mathcal{M}$ is base point free.

Non-commutative Mori contraction

Let $Y =$ nc smooth proj surface,
$M = K$-non-effective rational curve with $M^2 = 0$,
$\mathcal{M}/X$ comp of Hilb $\mathcal{O}_Y$ containing M.

Theorem (MAIN, C-Nyman)

If for every simple 0-dim quotient $P \in \operatorname{mod} Y$ of M we have $M.P = 0$, then $\mathcal{M}$ is base point free. There is an associated Fourier-Mukai morphism $f : Y \longrightarrow X$, called a nc Mori contraction.

Rem

Non-commutative Mori contraction

Let Y = nc smooth proj surface,
$M = K$-non-effective rational curve with $M^2 = 0$,
$\mathcal{M}/X$ comp of $\mathrm{Hilb}\,\mathcal{O}_Y$ containing M.

Theorem (MAIN, C-Nyman)

If for every simple 0-dim quotient $P \in \mathrm{mod}\,Y$ of M we have $M.P = 0$, then $\mathcal{M}$ is base point free. There is an associated Fourier-Mukai morphism $f : Y \longrightarrow X$, called a nc Mori contraction.

Rem

- Thm still holds with X replaced with finite cover.

Non-commutative Mori contraction

Let Y = nc smooth proj surface,
$M = K$-non-effective rational curve with $M^2 = 0$,
$\mathcal{M}/X$ comp of $\mathrm{Hilb}\,\mathcal{O}_Y$ containing M.

Theorem (MAIN, C-Nyman)

If for every simple 0-dim quotient $P \in \mathrm{mod}\,Y$ of M we have $M.P = 0$, then $\mathcal{M}$ is base point free. There is an associated Fourier-Mukai morphism $f : Y \longrightarrow X$, called a nc Mori contraction.

Rem

- Thm still holds with X replaced with finite cover.
- (Smith) $\exists$ weird eg of nc quadrics with a point lying on all lines of a ruling.

Non-commutative Mori contraction

Let Y = nc smooth proj surface,
$M = K$-non-effective rational curve with $M^2 = 0$,
$\mathcal{M}/X$ comp of Hilb $\mathcal{O}_Y$ containing M.

Theorem (MAIN, C-Nyman)

If for every simple 0-dim quotient $P \in \text{mod}\, Y$ of M we have $M.P = 0$, then $\mathcal{M}$ is base point free. There is an associated Fourier-Mukai morphism $f : Y \longrightarrow X$, called a nc Mori contraction*.*

Rem

- Thm still holds with X replaced with finite cover.
- (Smith) $\exists$ weird eg of nc quadrics with a point lying on all lines of a ruling.
- Recovers fibration $f : Y \longrightarrow X$ of nc ruled surface if $\mathcal{M}/X$ is family of ruling lines.

Thanks for your time

Auslander condition on a dualising complex

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.

Auslander condition on a dualising complex

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^{\vee} \in D^+(A^o \otimes_k A)$.

Auslander condition on a dualising complex

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^\vee \in D^+(A^o \otimes_k A)$.
This is an *Auslander balanced dualising complex* if the following conditions hold:

Auslander condition on a dualising complex

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^{\vee} \in D^{+}(A^{o} \otimes_k A)$.
This is an *Auslander balanced dualising complex* if the following conditions hold:

1. $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension.

Auslander condition on a dualising complex

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^\vee \in D^+(A^o \otimes_k A)$.
This is an *Auslander balanced dualising complex* if the following conditions hold:

1. $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension.
2. For every $i, j \in \mathbb{N}$ and noetherian module $M \in \mathrm{Gr}\, A$,

Auslander condition on a dualising complex

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^\vee \in D^+(A^o \otimes_k A)$.
This is an *Auslander balanced dualising complex* if the following conditions hold:

1. $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension.
2. For every $i, j \in \mathbb{N}$ and noetherian module $M \in \operatorname{Gr} A$, $(R^i\Gamma_{\mathfrak{m}} M)_{\geq j}$ is finite dimensional.

Auslander condition on a dualising complex

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^{\vee} \in D^+(A^o \otimes_k A)$.
This is an *Auslander balanced dualising complex* if the following conditions hold:

1. $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension.
2. For every $i, j \in \mathbb{N}$ and noetherian module $M \in \operatorname{Gr} A$, $(R^i\Gamma_{\mathfrak{m}} M)_{\geq j}$ is finite dimensional.
3. For any noetherian $M \in \operatorname{Gr} A$ and left A-submodule $N < \operatorname{Ext}^j_A(M, \omega_A)$ we have

Auslander condition on a dualising complex

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^\vee \in D^+(A^o \otimes_k A)$.
This is an *Auslander balanced dualising complex* if the following conditions hold:

1. $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension.
2. For every $i, j \in \mathbb{N}$ and noetherian module $M \in \operatorname{Gr} A$, $(R^i\Gamma_{\mathfrak{m}} M)_{\geq j}$ is finite dimensional.
3. For any noetherian $M \in \operatorname{Gr} A$ and left A-submodule $N < \operatorname{Ext}^j_A(M, \omega_A)$ we have $\operatorname{Ext}^i_{A^o}(N, \omega_A) = 0$ whenever $i < j$.

Auslander condition on a dualising complex

Let $A =$ noeth conn graded k-algebra fin gen in degree 1.
Let $\omega_A := R\Gamma_{\mathfrak{m}}(A)^\vee \in D^+(A^o \otimes_k A)$.
This is an *Auslander balanced dualising complex* if the following conditions hold:

1. $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension.
2. For every $i, j \in \mathbb{N}$ and noetherian module $M \in \operatorname{Gr} A$, $(R^i\Gamma_{\mathfrak{m}} M)_{\geq j}$ is finite dimensional.
3. For any noetherian $M \in \operatorname{Gr} A$ and left A-submodule $N < \operatorname{Ext}^j_A(M, \omega_A)$ we have $\operatorname{Ext}^i_{A^o}(N, \omega_A) = 0$ whenever $i < j$.
4. The above three conditions also hold for the opposite algebra A^o.