Algebraic stacks in the representation theory of finite-dimensional algebras

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Introduction

always work over base field k algebraically closed of char 0.

Motto

Moduli stacks are a fruitful way to study non-commutative algebra, because they are a machine to construct functors.

Plan of talk

- Recall the variety of representations of a quiver with relations.
- Brief user's guide to stacks in representation theory.

Question

Given a finite dimensional algebra A, how do you find an algebraic stack which is derived equivalent to it?

We finally,

- introduce a new moduli stack of "Serre stable representations", which gives a first approximation to answering this question.

Quivers and representations

We use the following notation

- quiver $Q = (Q_0 = \text{vertices}, Q_1 = \text{edges})$ without oriented cycles
- kQ the path algebra & $I \triangleleft kQ$ an admissible ideal of relations
- $M = \oplus_{v \in Q_0} M_v$ is a (right) $A = kQ/I$-module i.e. a representation of Q with relations I.
- The *dimension vector* of M is $\vec{\dim} M = (\dim_k M_v)_{v \in Q_0} \in \mathbb{Z}^{Q_0} \simeq K_0(A)$.

Representation variety

Let's classify representations with dim vector $\vec{d} = (d_v)$. Consider one such M.

- Picking bases i.e. isomorphisms $M_v \simeq k^{d_v}$ gives a unique point of

$$\mathrm{Rep}(Q, \vec{d}) := \prod_{v \to w \in Q_1} \mathrm{Hom}_k(k^{d_v}, k^{d_w}).$$

- Choice of basis is up to group $GL(\vec{d}) := \prod_{v \in Q_0} GL(d_v)$.
- If $I \neq 0$, then kQ/I-modules correspond to some closed subscheme

$$\mathrm{Rep}(Q, I, \vec{d}) \subseteq \mathrm{Rep}(Q, \vec{d}).$$

- $GL(\vec{d})$ acts on $\mathrm{Rep}(Q, I, \vec{d})$ and orbits correspond to isomorphism classes of modules (with dim vector $\vec{d}$),
- stabilisers correspond to automorphism groups of M.
- The diagonal copy of $k^\times$ acts trivially so $PGL(\vec{d}) := GL(\vec{d})/k^\times$ also acts.

Motivating example à la King

$Q =$ Kronecker quiver $v \rightrightarrows w$, $\quad \vec{d} = \vec{1} = (1 \quad 1)$.

$$k \overset{x}{\underset{y}{\rightrightarrows}} k \ \in \mathrm{Rep}(Q, \vec{1}) \simeq k^2 = \mathbb{A}^2$$

$PGL(\vec{1}) = k^{\times 2}/k^{\times} \simeq k^{\times}$ acts by scaling, so if we omit $(x, y) = (0, 0)$ (explain later) have quotient $(\mathrm{Rep}(Q, \vec{1}) - (0, 0))/PGL \simeq \mathbb{P}^1$.

We get a *family* of modules $M_{(x:y)} = \ M_{(x:y),v} \overset{x}{\underset{y}{\rightrightarrows}} M_{(x:y),w}$

parametrised by $(x : y) \in \mathbb{P}^1$ which gives "the" *universal representation*

$$\mathcal{U} = \ \mathcal{O}_{\mathbb{P}^1} \overset{x}{\underset{y}{\rightrightarrows}} \mathcal{O}_{\mathbb{P}^1}(1)$$

Interesting Fact

$\mathcal{U}$ is an $\mathcal{O}_{\mathbb{P}^1} - A$-bimodule whose dual ${}_A\mathcal{T}_{\mathcal{O}_{\mathbb{P}^1}} = \mathcal{H}om_{\mathbb{P}^1}(\mathcal{U}, \mathcal{O})$ induces inverse derived equivalences

$$\mathrm{RHom}_{\mathbb{P}^1}(\mathcal{T}, -) : D^b(\mathbb{P}^1) \longrightarrow D^b(A), \ - \otimes_A^L \mathcal{T} : D^b(A) \longrightarrow D^b(\mathbb{P}^1)$$

Stacks: via categorifying Grothendieck's functor of points

To generalise this eg, need to enlarge category of schemes. A scheme X is not determined by its k-points, but is determined by all its R-points (R comm ring). More precisely, it's determined by

Functor of points

the *functor of points* of X, which is the covariant functor
$h_X = \mathrm{Hom}_{\mathrm{Scheme}}(\mathrm{Spec}\,(-), X) : \mathrm{CommRing} \longrightarrow \mathrm{Set}$
so $h_X(R) = \{f : \mathrm{Spec}\ R \longrightarrow X\}$

Remark Compare with maximal atlas defn of a manifold.
We "categorify" this defn, and let Gpd be the category of groupoids = small categories with all morphisms invertible.

"Definition" (Stack)

A *stack* is a pseudo-functor $h : \mathrm{CommRing} \longrightarrow \mathrm{Gpd}$ + lots of axioms.

Think of the isomorphism classes of objects in the category $h(k)$ as the "k-points" & the category now remembers automorphisms.

Example: Stacky group quotients

Let G be an algebraic group acting on a k-variety X.

Want a "stacky" group quotient $[X/G]$ st "k-points" are the G-orbits $G.x$, & the automorphism group of such a point is $\mathrm{Stab}_G\, x < G$.

Recall A scheme morphism $\tilde{U} \longrightarrow U$ is a *G-torsor or G-bundle* if G acts on $\tilde{U}$ and trivially on U, is G-equivariant and locally on U is the trivial G-torsor $pr : G \times U \longrightarrow U$.

Motivation There should be a G-torsor $\pi : X \longrightarrow [X/G]$ so an object of $f \in [X/G](R)$ gives a Cartesian square

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\phi} & X \\ q\downarrow & & \downarrow\pi \\ U := \mathrm{Spec}\ R & \xrightarrow[f]{} & [X/G] \end{array}$$

$\Longrightarrow$ objects of $[X/G](R)$ are pairs (ϕ, q) st
$q : \tilde{U} \longrightarrow \mathrm{Spec}\ R$ is a G-torsor & $\phi : \tilde{U} \longrightarrow X$ is G-equivariant.

Effect of stabiliser groups

Define category of coherent sheaves $\mathrm{Coh}[X/G]$ = category of G-equivariant coherent sheaves on X e.g. if X smooth, $\omega_{[X/G]} := \omega_X$.

Consider case $X = \mathbb{A}^1_x$ & $G = \mu_p = \langle \zeta = \sqrt[p]{1} \rangle$ acts by multn, so action free on $x \neq 0$ but $\mathrm{Stab}_G\, 0 = \mu_p$.

k-**points** are parametrised by $y = x^p$.

- If $y \neq 0$ then $k[x]/(x^p - y)$ is a simple sheaf on $[X/G]$.
- If $y = 0$, then $k[x]/(x^p)$ is non-split extension of p non-isomorphic simples $k[x]/(x)$ with μ_p-action given by the p characters of μ_p.

General Fact

If $\tilde{U} \longrightarrow U$ is a G-torsor, then $[\tilde{U}/G] \simeq U$. Here $[(\mathbb{A}^1_x - 0)/\mu_p] \simeq \mathbb{A}^1_y - 0$.

- $\omega_X = k[x]dx$ & $\omega_{[X/G]} \otimes_{[X/G]} -$ permutes the simples with $x = 0$ cyclically.

Families through stacky points

Note there is also a "birational" map $[\mathbb{A}^1_x/\mu_p] \longrightarrow \mathbb{A}^1_y$. The rational inverse $\phi : \mathbb{A}^1_y - 0 \longrightarrow [\mathbb{A}^1_x/\mu_p]$ given by

$$\begin{array}{ccc} \mathbb{A}^1_x - 0 & \longrightarrow & \mathbb{A}^1_x \\ {\scriptstyle x \mapsto x^p = y}\big\downarrow & & \\ \mathbb{A}^1_y - 0 & & \end{array}$$

Important Phenomenon

You can't extend ϕ to all of $\mathbb{A}^1_y$, except by first passing to to an étale cover of $\mathbb{A}^1_y - 0$ as below.

Have tautological quotient map $\mathbb{A}^1_x \longrightarrow [\mathbb{A}^1_x/\mu_p]$ defined by

$$\begin{array}{ccc} \mu_p \times \mathbb{A}^1_x & \xrightarrow{\text{action}} & \mathbb{A}^1_x \\ {\scriptstyle pr}\big\downarrow & & \\ \mathbb{A}^1_x & & \end{array}$$

This process is called "stable reduction".

Weighted projective lines

Can define stacks via gluing just as for schemes.

Let $y_1, \dots, y_n \in \mathbb{P}^1$ and $p_1, \dots, p_n \geq 2$ be integer weights.
There is a stack $\mathbb{W} = \mathbb{P}^1(\sum p_i y_i)$ and map $\pi : \mathbb{P}^1(\sum p_i y_i) \longrightarrow \mathbb{P}^1$ which is

- an isomorphism away from the y_i,
- locally near y_i, it looks like $[\mathbb{A}^1_x / \mu_{p_i}] \longrightarrow \mathbb{A}^1_y$

We call $\mathbb{P}^1(\sum p_i y_i)$ a *weighted projective line*.

π^* induces an isomorphism

$$k^2 = \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}(1)) \longrightarrow \mathrm{Hom}_{\mathbb{W}}(\pi^* \mathcal{O}, \pi^* \mathcal{O}(1)).$$

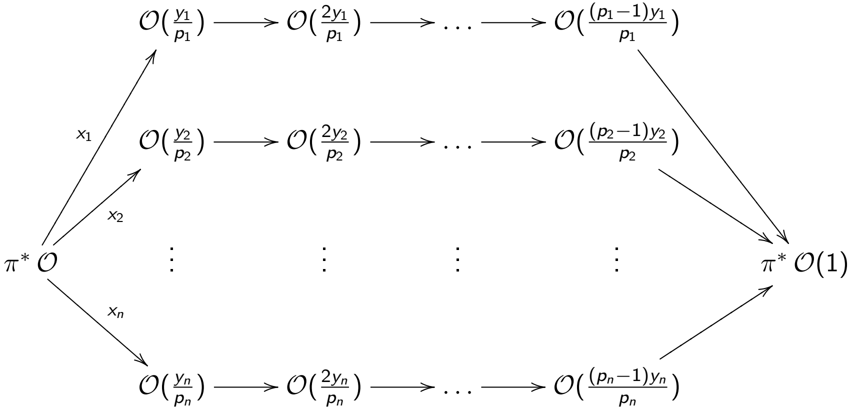
If $f_i \in \mathrm{Hom}_{\mathbb{W}}(\pi^* \mathcal{O}, \pi^* \mathcal{O}(1))$ corresponds to y_i, then

$$\mathrm{coker}\,(f_i : \pi^* \mathcal{O} \longrightarrow \pi^* \mathcal{O}(1))$$

is the non-split extension of p_i non-isomorphic simples on previous slide.

Canonical Algebra

Factorising f_i into p_i inclusions gives



Thm(Geigle-Lenzing) The above is a tilting bundle on $\mathbb{P}^1(\sum p_i y_i)$ with endomorphism ring the corresponding canonical algebra.

Moduli stack of isomorphism classes of $A = kQ/I$-modules

Fix dim vector $\vec{d} \in K_0(A)$. There's a stack $\text{Iso}(A, \vec{d})$ with k-points the iso classes of A-modules dim vector $\vec{d}$ & automorphisms = module automorphisms.
$\text{Iso}(A, \vec{d})(R)$ = category of (R, A)-modules $\mathcal{M} = \oplus \mathcal{M}_v$, with

- $\mathcal{M}_v$ loc free rank d_v/R,
- Morphisms = bimodule isomorphism

Important Facts

- $\text{Iso}(A, \vec{d}) \simeq [\text{Rep}(Q, I, \vec{d})/GL(\vec{d})]$.
- Tautologically, there is a universal A-module $\mathcal{U} = \oplus \mathcal{U}_v$ over $\text{Iso}(A, \vec{d})$.

Note These will never be weighted projective lines because all modules have $k^\times$ in their automorphism group!

Rigidified moduli stack of A-modules

We *rigidify* the stack to remove this common copy of $k^\times$. Define (when some $d_v = 1$ else need stackification)
$\mathrm{RigIso}(A, \vec{d})(R)$ has same objects as $\mathrm{Iso}(A, \vec{d})(R)$, but

- a morphism in $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ is an equivalence class of (R, A)-bimodule isomorphisms $\psi : \mathcal{M} \longrightarrow L \otimes_R \mathcal{N}$ where L is a line bundle on R,
- $\psi : \mathcal{M} \longrightarrow L \otimes_R \mathcal{N}, \psi' : \mathcal{M} \longrightarrow L' \otimes_R \mathcal{N}$ are equivalent if there's an iso $l : L \longrightarrow L'$ st $\psi' = (l \otimes 1)\psi$.

Important Facts

- $\mathrm{RigIso}(A, \vec{d}) \simeq [\mathrm{Rep}(Q, I, \vec{d})/PGL(\vec{d})]$.
- Tautologically, there is a universal A-module $\mathcal{U} = \oplus \mathcal{U}_v$ over $\mathrm{RigIso}(A, \vec{d})$, unique up to line bundle.

Serre functor map RigIso $- \to$ RigIso

Assume now $\mathrm{gl.\,dim}\, A < \infty$ & write $DA = \mathrm{Hom}_k(A, k)$.

Recall we have a Serre functor $\nu = - \otimes^L_A DA$ on $D^b_{fg}(A)$. Define $\nu_d = \nu \circ [-d]$.

Given a k-point of $\mathrm{RigIso}(A, \vec{d})$ i.e. A-module M, $\nu_d M$ may or may not define a k-point of $\mathrm{RigIso}(A, \vec{d})$.

Proposition

The locus of modules where it does, defines a locally closed substack $\mathrm{RigIso}(A, \vec{d})^0$ of $\mathrm{RigIso}(A, \vec{d})$. It is open if $d = \mathrm{pd}\, DA$ or $\mathrm{pd}\, DA - 1$.

We hence obtain a partially defined self-map

$$\nu_d : \mathrm{RigIso}(A, \vec{d})^0 \longrightarrow \mathrm{RigIso}(A, \vec{d})$$

The Serre stable moduli stack

The *Serre stable moduli stack* $\mathrm{RigIso}(A,\vec{d})^S$ is the fixed point stack i.e. fibre product

$$\begin{array}{ccc}
\mathrm{RigIso}(A,\vec{d})^S & \longrightarrow & \mathrm{RigIso}(A,\vec{d})^0 \\
\downarrow & & \downarrow \Gamma_{\nu_d} \\
\mathrm{RigIso}(A,\vec{d}) & \xrightarrow{\Delta} & \mathrm{RigIso}(A,\vec{d}) \times \mathrm{RigIso}(A,\vec{d})
\end{array}$$

The category of k-points $\mathrm{RigIso}(A,\vec{d})^S(k)$ has

- Objects: isomorphisms $M \xrightarrow{\sim} \nu_d M$ where M is an A-module dim vector $\vec{d}$
- Morphisms: diagrams of isomorphisms which commute up to scalar

$$\begin{array}{ccc}
M & \longrightarrow & \nu_d M \\
\theta \downarrow & & \downarrow \nu_d \theta \\
N & \longrightarrow & \nu_d N
\end{array}$$

Objects of $\mathrm{RigIso}(A,\vec{d})^S(R)$ are (R,A)-bimodule isomorphisms $\mathcal{M} \simeq L \otimes_R \mathcal{M} \otimes_A^L DA[-d]$, where L is a line bundle.

Serre stability alters points: eg Kronecker algebra

$Q =$ Kronecker quiver $v \rightrightarrows w$, $\vec{d} = \vec{1} = (1 \quad 1)$. $A = kQ, d = 1$.

$$M: \ k \underset{0}{\overset{0}{\rightrightarrows}} k$$

has a projective summand $0 \rightrightarrows k$ so $M \not\simeq \nu_1 M$
$\implies$ no corresponding point of $\mathrm{RigIso}(A, \vec{1})^S$.

However, for the universal representation

$$\mathcal{U} = \ \mathcal{O}_{\mathbb{P}^1} \underset{y}{\overset{x}{\rightrightarrows}} \mathcal{O}_{\mathbb{P}^1}(1)$$

we have $\mathcal{U} \otimes^L_A DA[-1] \simeq \omega_{\mathbb{P}^1} \otimes_{\mathbb{P}^1} \mathcal{U}$ & in fact

Proposition

$\mathrm{RigIso}(A, \vec{1})^S \simeq \mathbb{P}^1$.

A similar result holds for the Beilinson algebra derived equivalent to $\mathbb{P}^d$.

Serre stability alters automorphism groups

$A =$ canonical algebra of $\mathbb{P}^1(3y)$. Let $d = 1, \vec{d} = \vec{1}$.

$$M := \begin{array}{ccccc} & & k & \xrightarrow{0} & k & & \\ & \overset{0}{\nearrow} & & & & \overset{0}{\searrow} & \\ k & & & \xrightarrow{\quad 1 \quad} & & & k \end{array}$$

is the direct sum of a ν_1-orbit corresponding to the 3 simple sheaves at $y = 0$.

- automorphisms of M in RigIso are $(k^\times)^3/k^\times \simeq (k^\times)^2$.
- automorphisms of M in RigIso^S are μ_3!

Why

$$\begin{array}{ccc} M & \longrightarrow & \nu_1 M \\ {\scriptstyle \theta \in (k^\times)^3}\downarrow & & \downarrow{\scriptstyle \nu_d \theta} \\ M & \longrightarrow & \nu_1 M \end{array} \quad , \nu_d \theta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \theta$$

commutes up to scalar $\iff \theta$ is an e-vector of the permutation matrix.

The k-points of RigIso^S

Note ν_d induces a (shifted) Coxeter transformation on $K_0(A)$.

If $M \in \mathrm{mod}\, A$ is *Serre stable* in sense $M \simeq \nu_d M$, then $\vec{d} := \vec{\dim} M$ is fixed by ν_d. We say $\vec{d}$ is *Coxeter stable*.

Proposition

Let M be a Serre stable module with $\vec{\dim} M$ minimal Coxeter stable. If $\mathrm{End}_A M$ is semisimple then

- Any two isomorphisms $\theta : M \longrightarrow \nu_d M, \theta' : M \longrightarrow \nu_d M$ are isomorphic in RigIso^S.
- The automorphism group in $\mathrm{RigIso}^S(k)$ of any such θ is μ_p where $p =$ no. Wedderburn components of $\mathrm{End}_A M$.

Some theorems

Theorem (C.-Lerner)

Let $\mathbb{W}$ be a weighted projective line which is *Fano or anti-Fano* i.e. $\omega_{\mathbb{W}}^{\mp 1}$ is ample or equiv, is not tubular. Let

- $\mathcal{T} = \oplus \mathcal{T}_v$ be a basic tilting bundle on $\mathbb{W}$
- $A = \operatorname{End}_{\mathbb{W}} \mathcal{T}$.

Then $\operatorname{RigIso}(A, \vec{\dim}\mathcal{T})^S \simeq \mathbb{W}$ & $\mathcal{T}$ is dual to the universal representation.

Remark Higher dimensional versions hold.

Theorem (C.-Lerner)

Let $A =$ canonical algebra. Then $\operatorname{RigIso}(A, \vec{1})^S$ is a weighted projective line derived equivalent to A & the universal representation is dual to the tilting bundle given earlier.

- Abdelghadir-Ueda have also exhibited weighted projective lines as moduli spaces, but of enriched quiver representations.
- The proof of the derived equivalence is via Bridgeland-King-Reid theory and is independent of Geigle-Lenzing's.

Reminder on Bridgeland-King-Reid theory

Let $\mathbb{W}$ be a smooth weighted projective variety. Then the set Ω of simple sheaves is a *spanning class* for $\mathrm{Coh}\,\mathbb{W}$.

Let $\mathcal{T}$ be an $(\mathcal{O}_{\mathbb{W}}, A)$-bimodule for some fin dim algebra A which is left locally free &

$$F = \mathrm{RHom}_{\mathbb{W}}(\mathcal{T}, -) : D^b_c(\mathbb{W}) \longrightarrow D^b_{fg}(A)$$

Theorem(Bridgeland-King-Reid)

Suppose for all $\mathcal{S}, \mathcal{S}' \in \Omega$ we have

- $F : \mathrm{Ext}^i_{\mathbb{W}}(\mathcal{S}, \mathcal{S}') \longrightarrow \mathrm{Ext}^i_A(F\mathcal{S}, F\mathcal{S}')$ is an isomorphism, and
- $\nu(F\mathcal{S}) \simeq F(\omega_{\mathbb{W}} \otimes_{\mathbb{W}} \mathcal{S})$.

Then F is a derived equivalence.

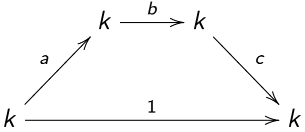
Remark Serre stability condition makes checking the 2nd condition easy.

A fresh look at the canonical algebra A

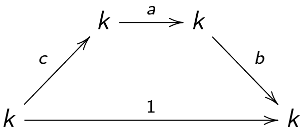
Step 1 Choose $\vec{d}$: For $\text{RigIso}^S \neq \varnothing$ need $\vec{d}$ fixed by Coxeter transformation $= \nu_1$ on $K_0(A)$. Use $\vec{d} = \vec{1}$ $\because$ it works and generates all such vectors if A is non-tubular.

Step 2 Compute Serre functor on some modules: eg for

$M :=$

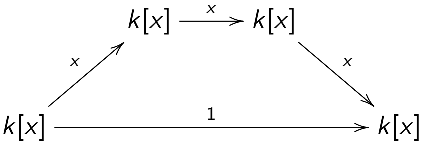


, $\nu_1 M :=$



Note iso class determined by product abc

Step 3 Guess a universal family/moduli space:



is a μ_3-equivariant family on $\mathbb{A}^1_x$. See $\text{RigIso}^S \simeq \mathbb{P}^1(3y)$.

Remark on stable reduction in this case

For $c \in k - 0$, we get a Serre stable family

$$M_c := \begin{array}{ccccc} & k & \xrightarrow{1} & k & \\ {\scriptstyle c}\nearrow & & & & \searrow{\scriptstyle 1} \\ k & & \xrightarrow{\quad\quad 1 \quad\quad} & & k \end{array}$$

which does not immediately extend to $c = 0$. Need first adjoin $\sqrt[3]{c}$ to get

$$M_{\sqrt[3]{c}} := \begin{array}{ccccc} & k & \xrightarrow{\sqrt[3]{c}} & k & \\ {\scriptstyle \sqrt[3]{c}}\nearrow & & & & \searrow{\scriptstyle \sqrt[3]{c}} \\ k & & \xrightarrow{\quad\quad 1 \quad\quad} & & k \end{array}$$

Extra Comments

- Method "works" because Serre stable moduli stack of "skyscraper sheaves" is the tautological moduli problem that recovers many stacks.
- Ideally we can apply Bridgeland-King-Reid theory to obtain independently many derived equivalences. Problem is we don't have many general results about the Serre stable moduli stack e.g. need a stable reduction theorem.
- For tame hereditary algebras, the preprojective algebra arises naturally in attempting to construct Serre stable objects.
- Case where you insert weights on intersecting divisors fails. Perhaps can be fixed by using the cotangent bundle.