A non-commutative Mori contraction

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Summary of axioms for nc (smooth proper) surface

Let Y = Grothendieck category $/k$ which is *strongly noetherian* i.e. Y_R is noeth for every noeth k-algebra R.

Y is a *nc surface* if

1. $\mathrm{Ext}^3_Y = 0$ and it's *Ext-finite* i.e. $\mathrm{Ext}^i_Y(M, \mathcal{N})$ f.g. $/R$ for $M \in \mathrm{mod}\, Y, \mathcal{N} \in \mathrm{mod}\, Y_R$.
2. it's *Gorenstein* i.e. there's an auto-equivalence $\omega_Y \otimes : \mathrm{mod}\, Y \longrightarrow \mathrm{mod}\, Y$ inducing Serre functor $\omega_Y \otimes (-)[2]$.
3. there's a *compatible* dimension function $\dim : \mathrm{mod}\, Y \longrightarrow \mathbb{N} \cup \{-\infty\}$.
4. there's 2-critical $\mathcal{O}_Y \in \mathrm{mod}\, Y$ giving *classical cohomology* $H^i(Y, -) := \mathrm{Ext}^i_Y(\mathcal{O}_Y, -)$.
5. it has *Halal Hilbert schemes & no shrunken flat deformations*.

Compatible Dimension Function

$\dim : \text{mod } Y \longrightarrow \mathbb{N} \cup \{-\infty\}$ is a *compatible dimension function* if it is *exact* i.e. for any exact seq in mod Y

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M''$$

we have $\dim M = \max\{\dim M', \dim M''\}$ & furthermore

it's compatible with the Serre functor in the sense that $\dim \omega_Y \otimes M = \dim M$.

Classical Cohomology

A 2-critical $\mathcal{O}_{\mathrm{Y}} \in \operatorname{mod} Y$ induces *classical cohomology* if for any 0-dim $P \in \operatorname{mod} Y$ we have

- $\operatorname{Ext}^1_Y(P, \mathcal{O}_{\mathrm{Y}}) = 0$ and,
- $H^0(P) \neq 0$.

Deformation properties

Y has *Halal Hilbert schemes* if for any $P \in \text{mod } Y$, there exists a Hilbert scheme $\text{Hilb } P$ parametrising quotients of P & $\text{Hilb } P$ is separated & a countable union of projective schemes which is locally of finite type.

Y has *no shrunken flat deformations* if for any flat family $\mathcal{M}$ of Y-modules / conn scheme of finite type X and closed points $x, y \in X$ such that there is an injective or surjective map $\phi : \mathcal{M} \otimes_X k(x) \longrightarrow \mathcal{M} \otimes_X k(y)$, we have that ϕ is an isomorphism.