A non-commutative Mori contraction

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Motivation: Commutative ruled surfaces

Fact Let Y be a comm smooth proj surface and $C \subset Y$ an extremal K-negative curve with $C^2 = 0$,
then there's a morphism $f : Y \longrightarrow X$ to a smooth proj curve X, contracting C, exhibiting Y as a ruled surface.

This is an example of a Mori contraction.

Motivating question for talk

Is there a nc version of this?

Need first define nc smooth proj surface etc.

NC projective schemes (Artin-Zhang)

Throughout work/ fixed alg closed field k of char 0.
Let $A =$ conn. graded k-algebra, fin gen in deg 1

Define $\text{Proj}\, A := \text{Gr}\, A/\text{tors}$

where $\text{Gr}\, A =$ category of graded A-modules

tors $=$ Serre sub-category of $A_{>0}$-torsion modules M

i.e. each element of M annihilated by some power of $A_{>0}$.

Why? (Serre) If A is homogeneous coord ring of proj scheme Y then

$\text{Proj}\, A \simeq$ category of quasi-coherent sheaves on Y

Additional hypotheses on A

$Y = \text{Proj } A$ is a *nc smooth proj d-fold* if further

1. A is a domain.
2. A is strongly noetherian i.e. $A \otimes_k R$ noeth for any comm noeth k-algebra R.
3. (Gorenstein) there are A-bimodules ω_Y, ω_Y^{-1} s.t. $\omega_Y[d+1]$ is an Auslander balanced dualising complex & $- \otimes_A \omega_Y, - \otimes_A \omega_Y^{-1}$ induce inverse auto-equivalences on Proj A.
4. (Smooth) $\text{Ext}_Y^{d+1}(-,-) = 0$. N.B. Mod Y has enough injectives.

Examples of nc smooth projective d-folds

- Skl_n, the n-dimensional Sklyanin algebra
- $Skl_4/(z)$ for generic $z \in Z(Skl_4)_2$
  nc quadric of Smith-Van den Bergh

Strongly noeth, Gorenstein hypotheses hold for A iff they hold for $A/(z)$ where z is homogeneous normal element of deg > 0.

Geometric techniques available (given hypotheses)

Let Y = nc smooth proj surface

Mod Y = category Y, mod Y = full subcat of noeth objects

Have following geometric concepts for Y

1. Cohomology
2. dimension function $\dim M, M \in \text{mod } Y$
3. Hilbert schemes
4. Bondal-Kapranov-Serre duality
5. Intersection theory

BUT not linear systems (yet??)

Cohomology (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

$$A =: \mathcal{O}_Y \in \text{mod } Y$$

For $M \in \text{Mod } Y$,

$$H^i(M) := \text{Ext}^i_Y(\mathcal{O}_Y, M)$$

Dimension of sheaves

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

Canonical dimension of noeth $M \in \text{Gr}\, A$ is

$$c.\dim M := \max\{i | \lim_n \text{Ext}^i_A(A/A_{>n}, M) \neq 0\}$$

If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ exact
then $c.\dim M = \max\{c.\dim M', c.\dim M''\}$
(by Yekutieli-Zhang)

On Y,

$$\dim = c.\dim -1$$

Hilbert schemes (Artin-Zhang)

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

For comm k-algebra R can define

$$\text{Mod } Y_R = \text{Proj } A_R = \text{Gr}(A \otimes_k R)/\text{tors}$$

For $\mathcal{M} \in \text{Mod } Y_R$ can define $\mathcal{M} \otimes_R - : \text{Mod } R \longrightarrow \text{Mod } Y$ & thus flatness $/R$.

Theorem (loosely stated Artin-Zhang 2001)

For $P \in \text{mod } Y$, there exists a Hilbert scheme $\text{Hilb } P$ parametrising quotients of P. $\text{Hilb } P$ is a countable union of projective schemes which is locally of finite type.

Bondal-Kapranov-Serre duality

Let $Y = \text{Proj } A =$ nc smooth proj d-fold

There are isomorphisms

$$\text{Ext}^i_Y(M, N) \simeq \text{Ext}^{d-i}_Y(N, M \otimes \omega_Y)^*$$

natural in $M, N \in \text{mod } Y$.

Note: $M = \mathcal{O}_Y$ is usual Serre duality.

Intersection theory (I.Mori-Smith)

Let $Y = \text{Proj } A =$ nc smooth proj surface

For $M, N \in \text{mod } Y$, have well-defined

$$M.N := -\sum_{i=0}^{2} (-1)^i \dim \text{Ext}^i_Y(M, N)$$

Why? If Y is comm, & $C, D \subset Y$ are curves

then $\mathcal{O}_C \,.\, \mathcal{O}_D = C.D$

New Questions

1. Find methods for constructing morphisms $f : Y \longrightarrow X$ between nc proj schemes i.e. adjoint functors $f^* : \mathrm{Mod}\, X \longrightarrow \mathrm{Mod}\, Y, f_* : \mathrm{Mod}\, Y \longrightarrow \mathrm{Mod}\, X$.
2. Suppose $Y =$ nc smooth proj surface with object like a K-negative curve with self-intersection 0. Can we use 1) to construct a "nc Mori contraction" $f : Y \longrightarrow X$ with X a smooth curve.
3. How much does such a nc Mori contraction behave like a commutative fibration?

K-negative rational curves with self-intersection 0

Let Y = nc smooth proj surface

Say $M \in \text{mod } Y$ is a *K-negative rational curve with self-intersection 0* if

1. M is a 1-critical quotient of $\mathcal{O}_Y$
2. $h^0(M) = 1, \ h^1(M) = 0, \ M^2 = 0$
3. $H^0(M \otimes \omega_Y) = 0$

N.B. If Y comm, $K.C < 0 \Longrightarrow H^0(\mathcal{O}_C \otimes \omega) = 0$.
Don't know if this is true nc.

Hilbert scheme of K-neg rat curve M with $M^2 = 0$

Let Y = nc smooth proj surface,

$M = K$-neg rat curve with $M^2 = 0$

Theorem (C-Nyman)

The point of $\mathrm{Hilb}\,\mathcal{O}_Y$ *corresponding to* M *is smooth and has 1-dim tangent space i.e. there's a projective curve* X*, smooth point* $p \in X$ *& flat family* $\mathcal{M}/X$ *of objects in* $\mathrm{mod}\,Y$ *with* $\mathcal{M} \otimes_X k(p) = M$.

Morphisms to curves

Let Y = nc smooth proj d-fold

X =(integral) proj curve

$\mathcal{M}/X$ = flat family of objects in mod Y

Wish to define exact $f^* : \text{Mod}\, X \longrightarrow \text{Mod}\, Y$ via "Fourier-Mukai transform"

$$\mathcal{L} \in \text{mod}\, X \rightsquigarrow \mathcal{M} \otimes_X \mathcal{L} \in \text{mod}\, Y_X$$

Want $f^*(\mathcal{L}) := \pi_*(\mathcal{M} \otimes_X \mathcal{L})$

for appropriate $\pi_* : \text{Mod}\, Y_X \longrightarrow \text{Mod}\, Y$

$\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$

Define π_* via relative Cech cohomology as usual i.e.

For $\mathcal{N} \in \text{Mod } Y_X$ & open affine cover $X = U \cup V$,

restriction maps give morphism in Mod Y

$$\mathcal{N}(U) \oplus \mathcal{N}(V) \xrightarrow{d} \mathcal{N}(U \cap V)$$

Define

$$\pi_*(\mathcal{N}) = \ker d$$

N.B.

1. π_* independent of choice of open affine cover
2. $f^* = \pi_*(\mathcal{M} \otimes_X -)$ is left exact

Q Is f^* right exact too?

Base point freedom

Defn Say $\mathcal{M}$ is *base point free* if for any simple $P \in \text{mod } Y$, $\text{Hom}_Y(M, P) = 0$ for a generic fibre $M \in \mathcal{M}$.

Theorem (C-Nyman)

If $\mathcal{M}/X$ is base point free then

- $f^* = \pi_*(\mathcal{M} \otimes_X -)$ *is exact so has a right adjoint.*
- f^* *preserves noeth objects.*

Stupid Eg Let $i : X \longrightarrow Y$ be an embedding of proj curve in a comm smooth proj surface,

$$\mathcal{M}/X = \mathcal{O}_X = \mathcal{O}_{\Gamma_i}$$

Then $f : Y \longrightarrow X$ has $f^* = i_*, f_* = i^!, f^*\mathcal{O}_X = \mathcal{O}_X$.

Non-commutative Mori contraction

Let Y = nc smooth proj surface.
M = K-neg rat curve with $M^2 = 0$
$\mathcal{M}\,/X$ comp of Hilb $\mathcal{O}_Y$ containing M.

Theorem (MAIN, C-Nyman)

If for every simple 0-dim quotient $P \in \text{mod}\, Y$ of M we have $M.P = 0$, then $\mathcal{M}$ is base point free. Say induced map $f : Y \longrightarrow X$ is a nc Mori contraction.

Rem

- Thm still holds with X replaced with finite cover.
- (Smith) $\exists$ weird eg of nc quadrics with a point lying on all lines of a ruling.
- Recovers fibration $f : Y \longrightarrow X$ of nc ruled surface if $\mathcal{M}\,/X$ is family of ruling lines.

New Questions

For a morphism of comm noeth integral schemes $f : Y \longrightarrow X$

- $f^* \mathcal{O}_X = \mathcal{O}_Y$
- $f_* \mathcal{O}_Y = \mathcal{O}_X$ if f projective with connected fibres & X normal.
- $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$ if f is a ruled surface.

Question

Do these hold for a nc Mori contraction?

Key Tool to answer question is
f flat $\Longrightarrow$ Leray spectral seq holds

$$\mathrm{Ext}^i_X(M, R^j f_* N) \Longrightarrow Ext^{i+j}_Y(f^* M, N)$$

Cohomology results

Let $f : Y \longrightarrow X$ be a nc Mori contraction as in main thm.
Nice case assume f *uniform* in sense $\forall M \in \mathcal{M}$

$$h^0(M) = 1, \; h^0(M \otimes \omega_Y) = 0.$$

In this case, X is smooth &
Facts

- $R^i f_*$ preserves noeth objects for all i.
- $R^i f_* = 0$ for $i > 1$ & $R^1 f_* \mathcal{O}_{\mathrm{Y}} = 0$.
- As $M \in \mathcal{M}$ varies, $r = h^1(M \otimes \omega_Y)$ is constant & $f_* \mathcal{O}_{\mathrm{Y}}$ is loc free of rank r.
- $f_* f^* \mathcal{O}_{\mathrm{X}}$ loc free of rank 1.

$\mathcal{O}_Y \longrightarrow f^* \mathcal{O}_X$

Let $f : Y \longrightarrow X$ be a uniform nc Mori contraction as in main thm.
Apply $\pi_* : \text{Mod } Y_X \longrightarrow \text{Mod } Y$ to universal quotient

$$\mathcal{O}_Y \otimes_X \mathcal{O}_X \longrightarrow \mathcal{M} \qquad \rightsquigarrow \qquad \nu : \mathcal{O}_Y \longrightarrow f^* \mathcal{O}_X$$

Theorem (C-Nyman)

- *Suppose the generic fibre of $\mathcal{M}$ is 1-critical & 1 fibre $M \in \mathcal{M}$ satisfies $H^1(M \otimes \omega_Y)^* = Ext^1_Y(M, \mathcal{O}_Y) \longrightarrow Ext^1_Y(M, M) = k$ is surjective. Then ν is injective & $h^1(M \otimes \omega_Y) = 1$.*
- *Suppose every $M \in \mathcal{M}$ is 1-critical & every simple quotient P of such an M has $h^0(P) = 1$. Then ν is an isomorphism.*

Future work

Our plan to establish a ruledness criterion for nc Mori contraction $f : Y \longrightarrow X$ is:

- Find $\mathcal{O}_{Y/X}(2)$ via $\omega_{Y/X}^{-1}$ or point scheme to show f is a conic bundle.
  Need nc relative Nakai-criterion. We have a version in point scheme case.
- Prove a nc version of Tsen's theorem.