A non-commutative Mori contraction

Daniel Chan reporting on
joint work with Adam Nyman

University of New South Wales

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Need first define nc smooth proj surface etc.

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Throughout work/ fixed alg closed field k of char 0.
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4. (Smooth) $\text{Ext}_Y^{d+1}(-,-) = 0$. N.B. Mod Y has enough injectives.

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Strongly noeth, Gorenstein hypotheses hold for A iff they hold for $A/(z)$ where z is homogeneous normal element of deg > 0.

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BUT not linear systems (yet??)

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On Y,

$$\dim = c.\dim -1$$

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Note: $M = \mathcal{O}_Y$ is usual Serre duality.

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2. Suppose $Y =$ nc smooth proj surface with object like a K-negative curve with self-intersection 0. Can we use 1) to construct a "nc Mori contraction" $f : Y \longrightarrow X$ with X a smooth curve.
3. How much does such a nc Mori contraction behave like a commutative fibration?

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Don't know if this is true nc.

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2. $f^* = \pi_*(\mathcal{M} \otimes_X -)$ is left exact

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$$\pi_*(\mathcal{N}) = \ker d$$

N.B.

1. π_* independent of choice of open affine cover
2. $f^* = \pi_*(\mathcal{M} \otimes_X -)$ is left exact

Q Is f^* right exact too?

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Then $f : Y \longrightarrow X$ has $f^* = i_*, f_* = i^!, f^* \mathcal{O}_X = \mathcal{O}_X$.

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- Recovers fibration $f : Y \longrightarrow X$ of nc ruled surface if $\mathcal{M}\,/X$ is family of ruling lines.

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$$\mathrm{Ext}^i_X(M, R^j f_* N) \Longrightarrow Ext^{i+j}_Y(f^* M, N)$$

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- *Suppose the generic fibre of $\mathcal{M}$ is 1-critical & 1 fibre $M \in \mathcal{M}$ satisfies* $$Ext^1_Y(M, \mathcal{O}_Y) \longrightarrow Ext^1_Y(M, M) \qquad \textit{is}$$ *surjective.*

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- Prove a nc version of Tsen's theorem.