Moduli stacks of Serre stable representations of finite-dimensional algebras

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Introduction

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Moduli stacks are a fruitful way to study a finite dimensional algebra A, because they are a machine to construct functors.

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Suggests should write A as a quotient of a

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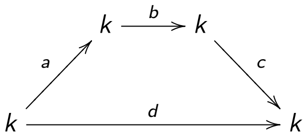

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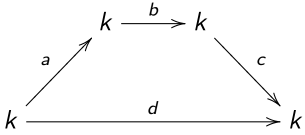

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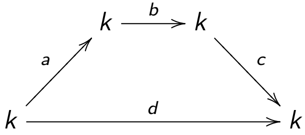

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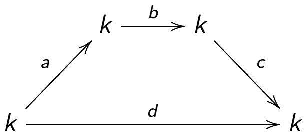

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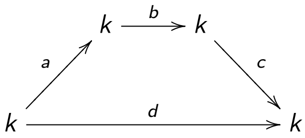

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Fact

Every fin dim algebra A is Morita equivalent to a quotient of kQ for Q a quiver with $\mathbb{Z}^{Q_0} \simeq K_0(A)$.

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- $\mathrm{Iso}_{rig}(A, \vec{d}) = [\mathrm{Rep}(Q, I, \vec{d})/PGL(\vec{d})]$ rigidified stack of isomorphism classes of dim vector $\vec{d}$ A-modules.

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NB Tilting theory usually approached starting with geometry & seeking a tilting bundle.

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Fact

Pic $\mathbb{W}$ is generated over Pic $\mathbb{P}^1$ by line bundles $\mathcal{O}(\frac{y_i}{p_i})$ subject to relations $\mathcal{O}(\frac{y_i}{p_i})^{\otimes p_i} \simeq \pi^* \mathcal{O}(y_i)$.

Geigle-Lenzing's theorem

$$
\begin{array}{ccccccccc}
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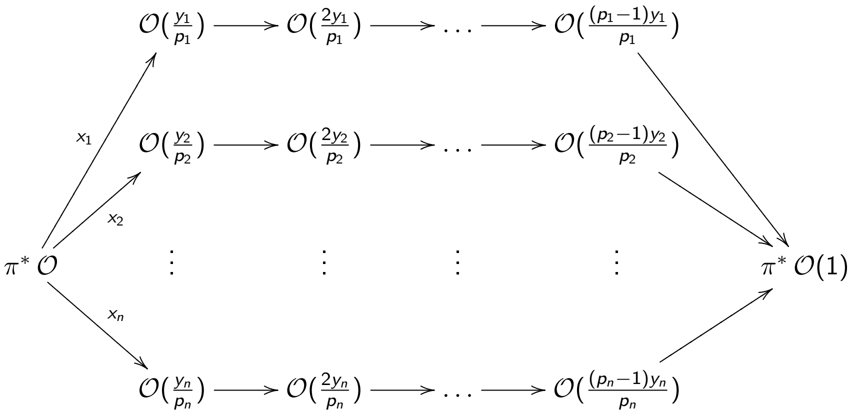

Thm(Geigle-Lenzing '87)

The above is a tilting bundle on $\mathbb{P}^1(\sum p_i y_i)$ with endomorphism ring Ringel's canonical algebra $A = kQ/I$. Here Q is as above & there are $n-2$ relations determined by $k^2 = \mathrm{Hom}_{\mathbb{W}}(\pi^* \mathcal{O}, \pi^* \mathcal{O}(1))$.

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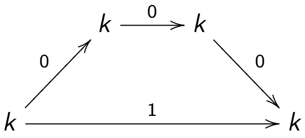
- Interesting obstacle: The stabiliser groups of $\mathbb{W}$ are finite cyclic groups whereas the automorphism groups of modules like

$M :=$



are typically products of $\mathbb{G}_m$ or similar such groups.

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We hence obtain a partially defined self-map

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It's instructive to consider analogously, the rigidified moduli stack $\mathbb{X}$ of finite length sheaves on a DM-stack $\mathbb{Y}$ and let $\mathbb{X}^S$ be the fixed point stack of $\omega_{\mathbb{Y}} \otimes_{\mathbb{Y}} -$.

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