Bimodule species and non-commutative projective lines

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$$M_0 \overset{x}{\underset{y}{\rightrightarrows}} M_1$$

given by right multiplication by $x, y \in V$.

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Indec in 3) given by $S^n(V) \overset{x}{\underset{y}{\rightrightarrows}} S^{n+1}(V)$ and dual, where S^n is n-th symmetric power.

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The geometric approach is based on Beilinson's derived equivalence which classically is . . .

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- For $n < 0$ have $\mathrm{Ext}^1(\mathcal{T}, \mathcal{O}(n)) : \ K^{-n} \rightrightarrows K^{-n-1}$

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Step 1 Define a non-commutative symmetric algebra $S = S^{nc}(V)$.

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where ϕ_i is the adjunction map.

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In fact, S has extremely nice homological properties. It is even Auslander regular in a certain sense.

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Quotient algebra $S/(g)$ is essentially a “twisted” $\mathbb{Z}$-indexed version of $F[x]$ where F is a degree δ extension of K.

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$\Lambda(V) \simeq \text{End}_{\text{coh } \mathbb{P}^{nc}(V)} \mathcal{T}$ and $R\text{Hom}(\mathcal{T}, -)$ induces a derived equivalence between $\mathbb{P}^{nc}(V)$ & $\Lambda(V)$.

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Can now define category of coherent sheaves on $\mathbb{P}^{nc}(V)$:

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Upshot The indec sheaves on $\mathbb{P}^{nc}(V)$ & indec $\Lambda(V)$-modules correspond.

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