Modular realisations of derived equivalences in representation theory

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Introduction

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Moduli stacks are a fruitful way to study a finite dimensional algebra A, because they are a machine to construct functors.

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Suggests should write A as a quotient of a

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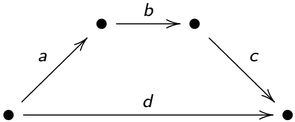

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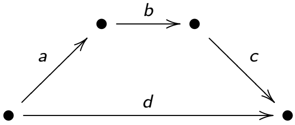

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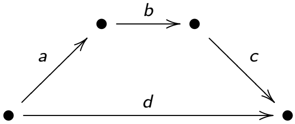

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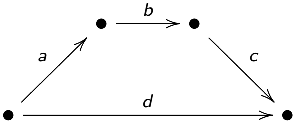

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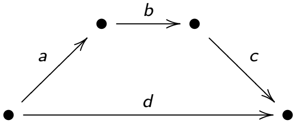

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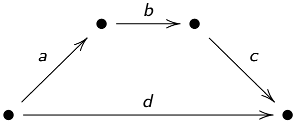

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$$\mathrm{RHom}_{\mathbb{P}^1}(\mathcal{T}, -) : D^b(\mathbb{P}^1) \longrightarrow D^b(A),\ - \otimes^L_A \mathcal{T} : D^b(A) \longrightarrow D^b(\mathbb{P}^1)$$

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A NO! because ...

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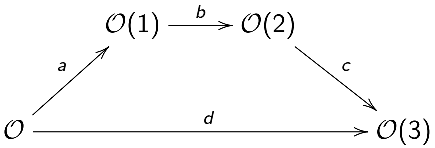

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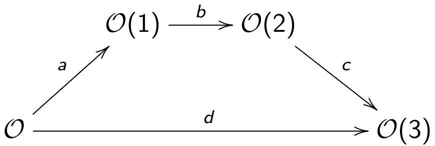

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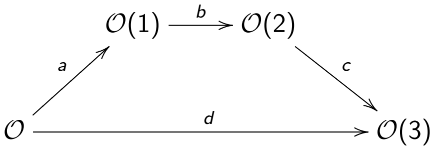

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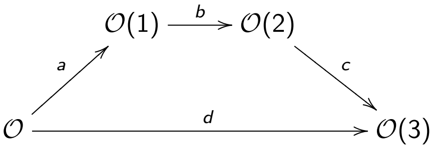

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Key to resolution remember some of the monoidal structure of $\mathrm{QCoh}(\mathbb{X})$.

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Thm-Defn (Abdelgadir-Ueda 2015)

The moduli stack $\mathbb{M}_{ref}$ of *refined representations* is the closed substack of $\tilde{\mathbb{M}}$ defined by the above constraints. This gives modular realisations of Geigle-Lenzing's derived equivalence's between orbifold projective lines and Ringel's canonical algebras.

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Rem The faithfulness condition forces the stabilisers of $\mathbb{X}$ to be abelian.

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Suppose further that there exist $\mathcal{L} \in \langle \mathcal{L}_1, \dots, \mathcal{L}_s \rangle$ such that $(\mathcal{T}, \mathcal{L})$ is ample. Then $\mathbb{X} \simeq \mathbb{M}^{L_1,\dots,L_s}$ & $\mathcal{T}^\vee$ is the universal sheaf.

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Thm (Abdelgadir-C.)

The above assignment defines an isomorphism of $\mathbb{M}^{L_1,\dots,L_s}$ onto an open substack of $\mathbb{M}_{ref}$.