Diagrams of linear maps and moduli spaces

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WARNING!! *WARNING!!* *NSFW*

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- For integer $d \geq 1$, a $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \ni z$ worth of indec. e.g. when $d = 1$

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- The *dimension vector* of a $\mathbb{C}\,Q$-module $M = \oplus M_v$ is vector $(\dim_{\mathbb{C}} M_v)_v \in \mathbb{Z}^{Q_0}$

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In particular, each isom class corresp to a unique pt of $\mathbb{P}^1$. We $\therefore$ say, $\mathbb{P}^1$ is the *moduli space* of $\mathbb{C}\,Q$-modules dim vector $\vec{1}$.

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- Indec sheaves $\mathcal{O}(d)$ corresp to $\mathbb{C}^d \rightrightarrows \mathbb{C}^{d+1}$ or $\mathbb{C}^{-d} \rightrightarrows \mathbb{C}^{-d-1}$.

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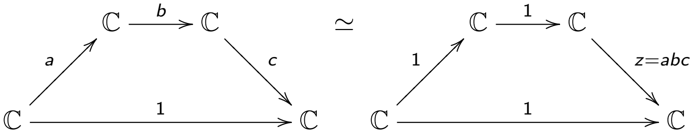

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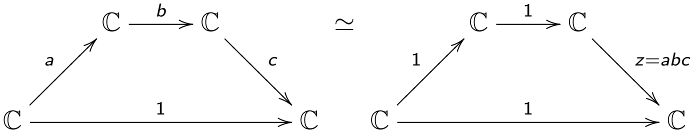

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Looks like the moduli space of modules dim vector $\vec{1}$ is again $\mathbb{P}^1 \ni z$.
BUT

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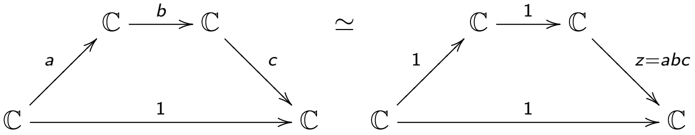

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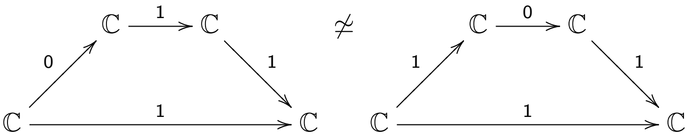

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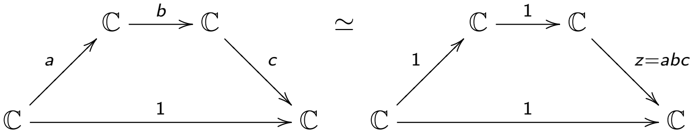

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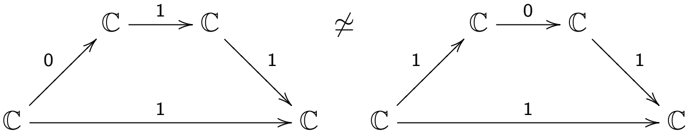

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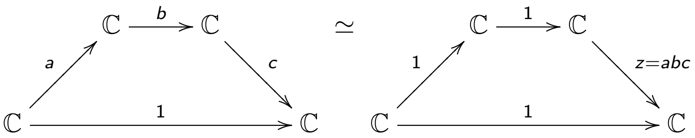

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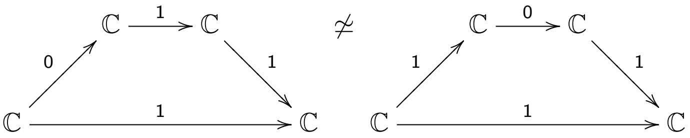

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It turns out we will only retain $a = b = c = 0$ when $abc = 0$.
Also, for this quiver Q, $\mathbb{C}\, Q$ not derived equivalent to $\mathbb{P}^1$.

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- Above theory generalises to A-modules, A a fin dim algebra, not nec a path algebra.

Modular realisation of derived equivalences

Consider a *type A canonical quiver*

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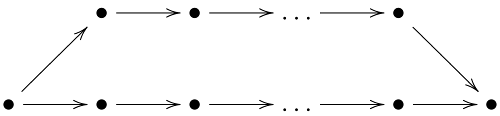

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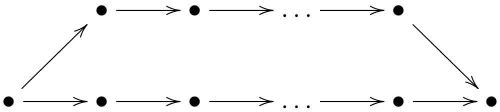

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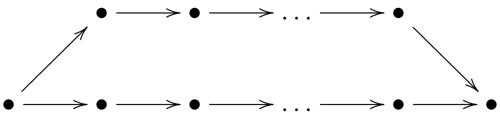

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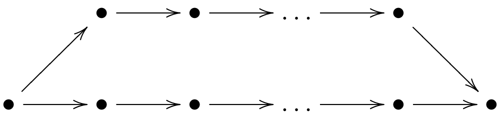

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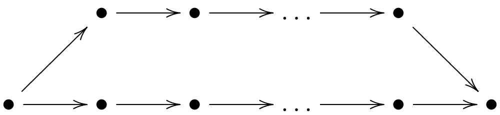

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$\mathbb{M}^S$ is an orbifold projective line i.e. a stack which looks like $\mathbb{P}^1$ but has stacky points which look locally like $[\mathbb{C}\,/\mu_n]$ at 0 (where stabiliser group is μ_n).

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- On scheme locus, just consider moduli of skyscraper sheaves. Our theory tells you how to handle stacky points.