Lecture 28: Lefschetz fixed point formula. Projective Varieties

Fixed-Point Formula

$X =$ smooth compact oriented man. dim d.

$\varphi : X \to X$, smooth map.

$\varphi^*|H^i(X,\mathbb{Q}) : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})$

Def$^{\underline{n}}$ 1: The Lefschetz number of f is

$$L(f) = \sum (-1)^i \operatorname{tr}\left(\varphi^* | H^i(X,\mathbb{Q})\right)$$

$$\qquad\qquad\qquad\qquad\quad \| $$

$$\qquad\qquad\qquad\qquad \operatorname{tr}\left(\varphi_* | H_i(X,\mathbb{Q})\right)$$

e.g. 1. $L(id) = \sum (-1)^i \underbrace{\operatorname{tr} id_{H^i(X,\mathbb{Q})}}_{\dim_{\mathbb{Q}} H^i(X,\mathbb{Q})}$

$= e(X)$.

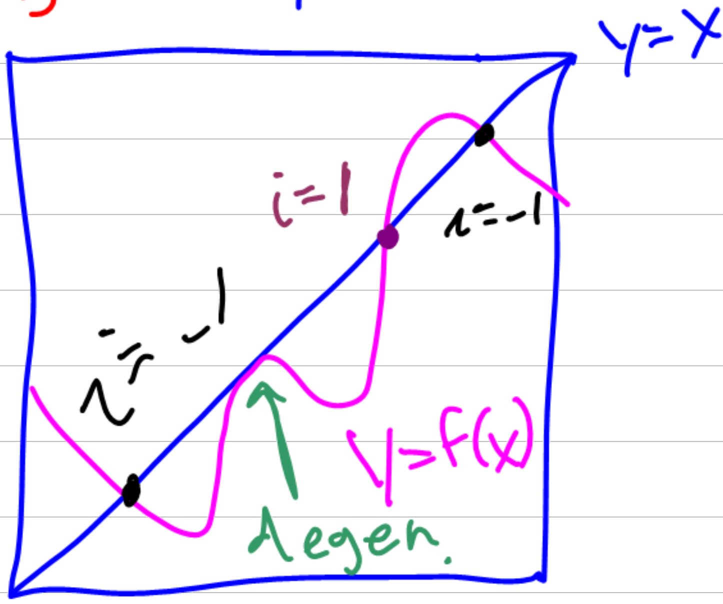
Def$^{\underline{n}}$ 2: A fixed pt $p \in X$ is non-degenerate if it is isolated &

$$0 \neq \det\left(id - d\varphi_p : T_pX \to T_pX\right)$$

In this case the index of p is

$$\iota_f(p) = \operatorname{sgn} \det\left(id - d\varphi_p\right)$$

E.g. 2. $\varphi: I \to X$



Thm: If the fixed points of $\varphi: X \to X$ are non-degenerate then

$$\sum_{p=\varphi(p)} i_\varphi(p) = L(\varphi).$$

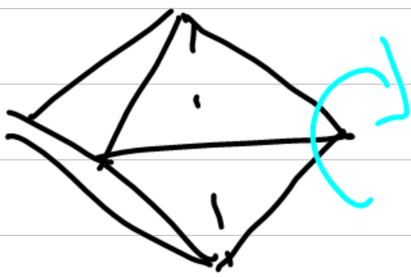
Heuristic Argument:

As for $e(x)$

Lemma: $L(f) = \sum (-1)^i \operatorname{tr}(\varphi^\# \mid C^p(x))$

Formula now natural guess if f simplicial & fixed points are vertices.

e.g.



For $p > 0$, φ moves every p-simplex $\Rightarrow$ $\varphi_\# \mid C_p(x)$ is permutation matrix with 0s on diagonal.

$\therefore \quad tr(\varphi_\# | C_p(X)) = 0.$

Sim.

$\varphi_\# | C_0(X)$ is permutation matrix with N = no. fixed pt is on diagonal. $\therefore \sum (-1)^p \, tr(\varphi_\# | C_p(X)) = N$ ∎

Projective Varieties

Let R = comm. ring

Def$^{\underline{n}}$ 3: $f(x_0, .., x_n) \in R[x_0, .., x_n]$ is homogeneous of degree d if every monomial term of f has degree d.

e.g. $3x^2 + 4xy + 7y^2$ is homog. but not $3x^2 + 4y$.

Easy Facts: If f is homog. deg d then

a For $\lambda \in R$, $f(\lambda x_0, .., \lambda x_n) = \lambda^d f(x_0, .., x_n)$

b If R = field & $\lambda \in R - 0$ then $\lambda(x_0, .., x_n)$ is a zero of f iff $(x_0, .., x_n)$ is.

Let $f_1, \dots, f_r \in \mathbb{Z}[x_0, \dots, x_n]$ be homog.
It can be viewed as a poly. $/\mathbb{F}$ for any field $\mathbb{F}$. The associated "projective variety$/\mathbb{Z}$" $V = V(f_1, \dots, f_r)$ has $\mathbb{F}$-rational points

$$V(\mathbb{F}) = \frac{\left\{ \underset{\sim}{\alpha} \in \mathbb{F}^{n+1} - \underset{\sim}{0} \;\middle|\; 0 = f_1(\underset{\sim}{\alpha}) = \dots = f_r(\underset{\sim}{\alpha}) \right\}}{\underset{\sim}{\alpha} \sim \lambda \underset{\sim}{\alpha}}$$

Write $V: f_1 = f_2 = \dots = f_r = 0 \subset \mathbb{P}^n$
& $(\alpha_0 : \alpha_1 : \dots : \alpha_n)$ for equiv. class of $\underset{\sim}{\alpha}$ in $V(\mathbb{F})$

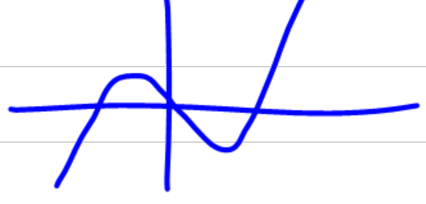
Eg. Elliptic curve

$$E: \; y^2 z = x^3 - x z^2 \subset \mathbb{P}^2$$

Q What's $E(\mathbb{F})$?
Let $(X : Y : Z) \in E(\mathbb{F})$
If $Z \neq 0$ can scale so $Z = 1$.
$\therefore$ Seek solns to

$$Y^2 = X^3 - X$$

$\mathbb{F} = \mathbb{R}$:



$\mathbb{F} = \mathbb{F}_2$: $X^3 - X = 0 \Rightarrow Y = 0$.

2 sol$\underline{\text{n}}$s : $(0:0:1)$, $(1:0:1)$

If $2=0$ then $X^3 = 0 \Rightarrow X = 0 \Rightarrow Y \neq 0$.

One extra soln at ∞ : $(0:1:0)$.

Rem: a $E(\mathbb{C}) \overset{\text{homeo}}{\simeq}$ 2-torus so is a topological group.

b $E(\mathbb{F}_{p^s})$ p prime, $s \in \mathbb{Z}_+$ is also an important group in cryptography if $p \neq 2$.

c $E(\overline{\mathbb{F}_p})$ is a countable set with cofinite topology.