MATH5665: Algebraic Topology- Course notes

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Abstract

These are the lecture notes for an Honours course in algebraic topology. They are based on standard texts, primarily Munkres's "Elements of algebraic topology" and to a lesser extent, Spanier's "Algebraic topology".

1 What's algebraic topology about?

Aim lecture: We preview this course motivating it historically.

Recall that a continuous map $f : X \longrightarrow Y$ of topological spaces is a *homeomorphism* if it is bijective and f^{-1} is also continuous. In this case we say X and Y are *homeomorphic* and write $X \simeq Y$.

Major Q in topology is how to determine if two spaces X, Y are homeomorphic or not.

A depends on whether they are homeomorphic or not. If they are, one usually guesses the homeomorphism. If not, one needs to be a bit more sophisticated. One usually uses *invariants* to distinguish them. Historically, the first example is below.

Euler Characteristic

Notation 1.1 *We denote the n-dim* unit ball *by* $B^n := \{\vec{v} \subset \mathbb{R}^n \,||\vec{v}| \leq 1\}$ *and the n-dim* unit sphere *by* $S^n := \{\vec{v} \subset \mathbb{R}^{n+1} \,||\vec{v}| = 1\}$

Consider a regular polyhedron $K \subset \mathbb{R}^3$ or more generally, any polyhedron homeomorphic to the unit sphere S^2. Let $V =$ no. vertices, $E =$ no. edges and $F =$ no. faces.

Theorem 1.2 (Euler) $V - E + F = 2$.

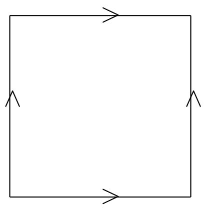
E.g. Check the cube, tetrahedron etc.

The quantity $e(K) := V - E + F$ is called the *Euler number* of K.

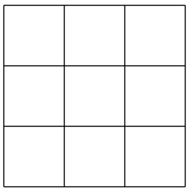
Quotient spaces

Let $X =$ topological space and $\sim$ be an equivalence relation on the underlying set so there is a set map $\pi : X \longrightarrow X/\sim : x \mapsto [x]$ where $X/\sim$ is the set of equivalence classes $[x]$. We put the strongest topology on $X/\sim$ so that π is continuous, viz, $U \subset X/\sim$ is open iff $\pi^{-1}(U)$ is open. The resulting topological space is called the *quotient space*.

E.g. Let $I = [0, 1] \subset \mathbb{R}$ and $X = I \times I$. We put the weakest equivalence relation on X s.t. $(0, x) \sim (1, x), (x, 0) \sim (x, 1)$ for $x \in I$. We sometimes sum up this info in the following picture:



Then $X/\sim$ is homeomorphic to the 2-torus $\mathbb{T}^2 = S^1 \times S^1$. Consider the following polyhedron K which is homeomorphic to $\mathbb{T}^2$:



Note $e(K) = 9 - 18 + 9 = 0 \neq 2$!

Upshot $\mathbb{T}^2$ is not homeomorphic to S^2.

Topological invariants

The Euler number is an example of a *topological invariant*, that is, an object associated to a top space that doesn't change when you pass to an homeomorphic space. Such invariants are often algebraic in nature, and algebraic topology is about the use of such invariants to show spaces are non-homeomorphic and deduce other interesting topological facts.

In this case, we will closely examine this Euler number, refining it to certain invariants called *Betti numbers* and more conceptually, to *homology groups*.

Phenomenon of independence of choice

The Euler number (as defined above) of a topological space requires choosing a polyhedron homeomorphic to it. There are many possibilities and the surprise is that it is independent of choice. This phenomenon is typical in the theory of homology and is something we will examine and study in depth.

Proof of Euler's thm

Actually, Euler's thm is not so hard to prove. Use induction. Maybe do next time as plenty of time.

ex.

2 Simplices

Aim Lecture We introduce simplices which are the building blocks for many topological spaces.

Barycentric co-ordinates

Notation 2.1 *Given a list $a_0, \dots, a_n$, we let $a_0 \dots, \widehat{a_i}, \dots, a_n$ to denote the same list except with a_i omitted.*

Proposition-Definition 2.2 *Let $a_0, \dots, a_n \in \mathbb{R}^N$ and $H \subset \mathbb{R}^{n+1}$ be the hyperplane $H := \{\vec{t} = (t_0, \dots, t_n) : \sum t_i = 1\}$. TFAE:*

i. $a_0 - a_i, \dots, \widehat{a_i - a_i}, \dots, a_n - a_i$ is linearly independent.

ii. The map $\beta : H \longrightarrow \mathbb{R}^N : \vec{t} \mapsto \sum_j t_j a_j$ is injective.

In this case, we say that $a_0, \dots, a_n$ are geometrically independent *and $\vec{t}$ are the* barycentric co-ordinates *of $\sum t_j a_j \in im\ \beta$.*

Proof. Note that $\vec{s} \in H \implies s_i = 1 - \sum_{j \neq i} s_j$. Hence $\beta(\vec{s}) = \beta(\vec{t})$ iff

$$0 = \sum_{j \neq i}(s_j - t_j)a_j + (-\sum_{j \neq i} s_j + \sum_{j \neq i} t_j)a_i = \sum_{j \neq i}(s_j - t_j)(a_j - a_i)$$

□

E.g. 3 collinear points are geom dependent.

E.g. Points with barycentric co-ords $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Regions defined by $t_2 = 0, < 0$.

Simplices

Definition 2.3 *Let $a_0, \ldots, a_n \subset \mathbb{R}^N$. The* span *of $a_0, \ldots, a_n \subset \mathbb{R}^N$ is*

$$\sigma = \langle\, a_0 \ldots a_n \,\rangle := \{\sum t_j a_j | \sum_j t_j = 1, t_j \geq 0\}.$$

The span is thus the convex hull. If further $a_0, \ldots, a_n$ are geom independent, then we say $\langle\, a_0 \ldots a_n \,\rangle$ is a (geometric) n-simplex *since then im β is n-dimensional.*

A (resp proper) *face* of the simplex $\sigma = \langle\, a_0 \ldots a_n \,\rangle$ is any simplex spanned by a (resp proper) subset of $\{a_0, \ldots, a_n\}$.

E.g.

The *i-th face* of $\langle\, a_0 \ldots a_n \,\rangle$ is $\langle\, a_0 \ldots \widehat{a_i} \ldots a_n \,\rangle$.

Proposition-Definition 2.4 *Let $\langle\, a_0 \ldots a_n \,\rangle$ be a simplex and $b_0, \ldots, b_n \in \mathbb{R}^M$.*

i. We have a well-defined map $f : \langle\, a_0 \ldots a_n \,\rangle \longrightarrow \langle\, b_0 \ldots b_n \,\rangle : \sum_j t_j a_j \mapsto \sum t_j b_j$. We call it affine linear *and denote it $l(\langle\, a_0 \ldots a_n \,\rangle, \langle\, b_0 \ldots b_n \,\rangle)$.*

ii. In particular, any two n-simplices are homeomorphic.

Proof. Note f is well-defined being the composite of β^{-1} with $\vec{t} \mapsto \sum t_j b_j$. Also (ii) holds since $l(\langle\, a_0 \ldots a_n \,\rangle, \langle\, b_0 \ldots b_n \,\rangle), l(\langle\, b_0 \ldots b_n \,\rangle, \langle\, a_0 \ldots a_n \,\rangle)$ are inverse continuous maps. □

Proposition 2.5 *An n-simplex σ is homeomorphic to B^n and its boundary (- union of its proper faces) is homeomorphic to S^{n-1}.*

Proof. This is an ex. Pictorially the argument is □

Topology of polytopes

A *polytope* X is a finite union of simplices $\sigma_1, \dots, \sigma_r \in \mathbb{R}^N$ such that any non-empty intersection $\sigma_i \cap \sigma_j$ is a face of both σ_i, σ_j for all i, j. Note X is compact since each σ_i is.

E.g.

The next "gluing lemma" allows one to glue locally defined continuous maps to obtain a global one.

Lemma 2.6 *Let top space X be a finite union $X_1 \cup \ldots \cup X_r$ of closed subsets X_i.*

i. $Y \subset X$ is closed (resp open) in X iff every $Y \cap X_i$ is closed (resp open) in X_i.

ii. If $f_i : X_i \longrightarrow Z$ are continuous maps for all i such that

(*) f_i, f_j agree on $X_i \cap X_j$ for all i, j.

Then there is a global continuous map $f : X \longrightarrow Z$ which restricts to each f_i.

Proof. (i) If $Y \cap X_i$ is closed in X_i, then it is closed in X. Hence $Y = \cup_i (Y \cap X_i)$ is closed too.

(ii) Condition (*) ensures that the f_i glue to a set map $f : X \longrightarrow Z$, whilst (i) ensures that it's continuous. □

3 Simplicial complexes and triangulation

Aim lecture Introduce simplicial complexes to describe how to glue simplices together to form interesting topological spaces.

Simplicial complex

Below, we let $\mathcal{P}'(S)$ to denote the power set of S less the element $\varnothing$. Let A be a finite ordered set whose elements we think of as vertices.

Definition 3.1 *A* simplicial complex K *(with vertices in A) consists of a subset K of $\mathcal{P}'(A)$ such that*

(*) if $\sigma \in K$ and $\varnothing \neq \tau \subset \sigma$ then $\tau \in K$.

We write the elements $\sigma \in K$ in the form $\sigma = a_0 \ldots a_d$ where $a_0 < \ldots < a_d$ are the elements or vertices *of σ and refer to σ as an* (abstract) d-simplex of K. *A* face of σ *is any non-empty subset of σ.*

A subcomplex K' *of K is any subset satisfying (*).*

e.g.

Proposition-Definition 3.2 *i. For $p \in \mathbb{N}$, the* p-skeleton *of K is the subset $K^{(p)}$ consisting of all d-simplices with $d \leq p$. It is a subcomplex of K.*

ii. For $\sigma \in K$, the subset $K_\sigma := \mathcal{P}'(\sigma)$ is a subcomplex of K called the complex associated to σ.

Proof. Easy. □

Defn $K^{(0)}$ is the set of vertices of K.

Polytope of a simplicial complex

Let K be a complex with $K^{(0)} = A$. Let $\mathbb{R}^A$ = set of functions $A \longrightarrow \mathbb{R}$ which is naturally a top space on identifying $\mathbb{R}^A$ with $\mathbb{R}^{|A|}$. Let $\varepsilon_a \in \mathbb{R}^A$ be the standard basis vector with $\varepsilon_a(a) = 1, \varepsilon_a(a') = 0$ if $a' \neq a$. Note that the ε_a are geometrically independent.

The following shows how a simplicial complex determines the combinatorial data to glue simplices together and form a topological space.

Proposition-Definition 3.3 *The* polytope $|K|$ of K *is the union in $\mathbb{R}^A$ of the following simplices: for each abstract simplex $\sigma = a_0 \dots a_d \in K$, the geometric simplex $\langle\, \sigma \,\rangle := \langle\, \varepsilon_{a_0} \dots \varepsilon_{a_d} \,\rangle$.*

i. $|K|$ is a polytope.

ii. $\varnothing \neq \tau \subseteq \sigma \Longrightarrow \langle\, \tau \,\rangle$ is a face of $\langle\, \sigma \,\rangle$ so $|K_\sigma| = \langle\, \sigma \,\rangle$.

iii. In fact, if we write $K = K_{\sigma_1} \cup \dots \cup K_{\sigma_r}$ for some $\sigma_i \in K$, then $|K| = |K_{\sigma_1}| \cup \dots \cup |K_{\sigma_r}|$ and $|K_{\sigma_i}| \cap |K_{\sigma_j}| = |K_{\sigma_i \cap \sigma_j}|$ (if we define $|K_\varnothing| = \varnothing$).

Proof. (ii) is clear whilst (iii) $\Longrightarrow$ (i). Finally, (iii) is easy (look at which barycentric co-ords are 0) and clear from any example. □

E.g. $K = K_{abc} \cup K_{bcd} = \{a, b, c, ab, bc, ac, abc, d, bd, cd, bcd\}$.

Example 3.4 *Sphere.*

Let $K = K_\sigma$ where σ is an n-simplex so $|K_\sigma| \simeq B^n$. Then $|K_\sigma^{(n-1)}|$ = boundary of $\langle\, \sigma \,\rangle$ so $|K_\sigma^{(n-1)}| \simeq S^{n-1}$.

Example 3.5 $K = K_{abcd}$.

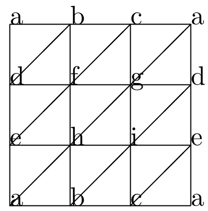
As we've seen, $|K^{(2)}_{abcd}|$ is homeomorphic to the 2-sphere. The 1-skeleton $|K^{(1)}_{abcd}|$

Triangulation

Definition 3.6 *A* triangulation *of a top space X is a simplicial complex K and homeomorphism $f : |K| \longrightarrow X$.*

Below, we use *labelled surface diagrams* to describe triangulations of well-known surfaces. Here K consists of 2-simplices given by the triangles below and all other simplices are their subsets.

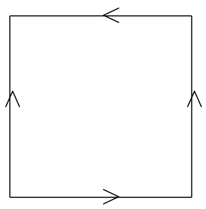
Example 3.7 *2-torus $\mathbb{T}^2$.*



The gluing lemma 2.6 shows there is indeed a triangulation of $\mathbb{T}^2$. Indeed, we identify $\mathbb{T}^2$ with $I \times I / \sim$ as in lecture 1 so the geometric simplices like $\langle\, abd \,\rangle$ above can be thought as subsets of $I \times I$ and hence in fact also of $\mathbb{T}^2$. Then the affine linear maps such as $l(\langle\, \varepsilon_a\, \varepsilon_b\, \varepsilon_d \,\rangle, \langle\, abd \,\rangle)$ glue together to an homeomorphism $|K| \longrightarrow \mathbb{T}^2$.

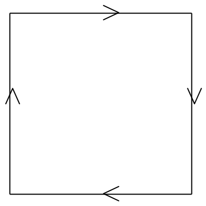
Note you can't omit a row!!

Example 3.8 *Klein bottle*



Example 3.9 *(real) Projective plane $\mathbb{R}\,P^2$.*

Be careful!



4 Homology groups

Aim lecture Intro homology groups which we'll see later give topological invariants.
K = simplicial complex.

p-chains

Definition 4.1 *The* group of p-chains $C_p(K)$ *of* K*, is the free abelian group on the set of* p*-simplices in* K*, i.e.*

$$C_p(K) = \{\sum_{\sigma} n_\sigma \, \sigma \, | n_s \in \mathbb{Z}, \sigma \in K \textit{ is a p-simplex}\}$$
$$=: \bigoplus_{\sigma \in K \textit{ is a p-simplex}} \mathbb{Z}\, \sigma$$

Its elements are called p-chains*. By default* $C_p(K) = 0$ *if* $p < 0$ *or there are no* p*-simplices.*

Given a p-simplex $a_o \dots a_p$ and a permutation $\pi \in \mathrm{Perm}\{0, \dots, p\}$, we let

$$[a_{\pi(0)} \dots a_{\pi(p)}] = \mathrm{sgn}(\pi) a_0 \dots a_p.$$

These are called *oriented* p*-simplices.* The idea is that the ordering on $K^{(0)}$ gives an *orientation* on all the simplices.
e.g. If $p = 1$ then $[ab] \neq [ba]$ can be represented by

$$a \longrightarrow b \;\text{ and }\; a \longleftarrow b$$

.

Similarly, for 2-simplices

Boundary operator
We define a group hom $\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$ by

$$\partial_p(a_0 \dots a_p) = \sum_{i=0}^{p} (-1)^i a_0 \dots \hat{a}_i \dots a_p$$

and extending linearly. We call it the p-th *boundary operator.*

Proposition 4.2

$$(*) \qquad \partial_p[a_0 \dots a_p] = \sum_{i=0}^{p} (-1)^i [a_0 \dots \hat{a}_i \dots a_p].$$

Proof. Note that (*) holds by definition when the a_i are in order. In general, we can use a permutation π to put the a_i in order. Now π is a product of, say ℓ tranpositions of form $(jj+1)$ so it suffices by induction on ℓ to show that the RHS of (*) changes sign whenever you swap a_j with a_{j+1}. We leave this as an exercise. □

The choice of orientation in ∂_p is best seen in an

e.g. $\partial_2(abc) = bc - ac + ab = [bc] + [ca] + [ab]$. Pictorially,

Note $\partial_1\partial_2(abc) = (b-c) + (c-a) + (a-b) = 0$!!

Homology groups

Lemma 4.3 $\partial_{p-1}\partial_p = 0$.

Proof.

$$\begin{aligned}\partial_{p-1}\partial_p(a_0 \dots a_p) &= \partial_{p-1}\sum_i (-1)^i a_0 \dots \hat{a}_i \dots a_p \\ &= \sum_{j<i}(-1)^{i+j}a_0 \dots \hat{a}_j \dots \hat{a}_i \dots a_p + \sum_{j>i}(-1)^{i+j-1}a_0 \dots \hat{a}_i \dots \hat{a}_j \dots a_p\end{aligned}$$

Note that the (i,j) term cancels with the (j,i)-term. □

We define *p-cycles* and *p-boundaries* to be resp, elements of the groups

$$Z_p(K) = \ker(\partial_p : C_p(K) \longrightarrow C_{p-1}(K)) \leq C_p(K), \quad B_p(K) = \mathrm{im}\,(\partial_{p+1} : C_{p+1}(K) \longrightarrow C_p(K)) \leq C_p(K).$$

Lemma $\implies \partial_p(B_p) = 0 \implies B_p(K) \leq Z_p(K)$. We define the p-th *homology group* to be

$$H_p(K) := Z_p(K)/B_p(K).$$

e.g. Consider the simplicial complex $K = K_{ab} \cup K_c$. DRAW $|K|$.

Consider the sequence of boundary maps

$$C_2(K) = 0 \longrightarrow C_1(K) = \mathbb{Z}\,ab \xrightarrow{\partial_1} C_0(K) = \mathbb{Z}\,a \oplus \mathbb{Z}\,b \oplus \mathbb{Z}\,c \xrightarrow{\partial_0} C_{-1}(K) = 0.$$

Note $B_0(K) = \mathrm{im}\,\partial_1 = \mathbb{Z}\,\partial_1(ab) = \mathbb{Z}(b-a)$ while $Z_0(K) = \ker\partial_0 = C_0(K) = \mathbb{Z}\,a \oplus \mathbb{Z}\,b \oplus \mathbb{Z}\,c$. Modulo B_0 we find $a \equiv b$ so $Z_0(K) = (\mathbb{Z}\,a \oplus \mathbb{Z}\,c) + B_0(K)$. Also $(\mathbb{Z}\,a \oplus \mathbb{Z}\,c) \cap B_0(K) = 0$. Hence the subgroup isomorphism theorem tells us that

$$H_0(K) = Z_0(K)/B_0(K) = [(\mathbb{Z}\,a \oplus \mathbb{Z}\,c) + B_0(K)]/B_0(K) \simeq (\mathbb{Z}\,a \oplus \mathbb{Z}\,c)/[(\mathbb{Z}\,a \oplus \mathbb{Z}\,c) \cap B_0(K)] \simeq \mathbb{Z}\,a \oplus \mathbb{Z}\,c.$$

The argument above suggests the following

Proposition 4.4 *For a simplicial complex K, $H_0(K) \simeq \mathbb{Z}^n$ where n is the no. connected components of $|K|$.*

Proof. See problem sheets. $\square$

5 Examples

Aim lecture We compute some examples illustrating how homology detects holes.

Circle

Example 5.1 $K = K^{(1)}_{abc}$

The $C^p(K)$ and boundary maps are:

$$0 = C_2(K) \xrightarrow{\partial_2} C_1(K) = \mathbb{Z}\, ab \oplus \mathbb{Z}\, bc \oplus \mathbb{Z}\, ac \xrightarrow{\partial_1} C_0(K) = \mathbb{Z}\, a \oplus \mathbb{Z}\, b \oplus \mathbb{Z}\, c \xrightarrow{\partial_0} C_{-1}(K) = 0$$

where

$$\partial_1 : [ab] = ab \mapsto b - a, \quad [bc] \mapsto c - b, \quad [ca] = -ac \mapsto a - c$$

Note if $p \neq 0$ or 1, then $Z_p(K) \leq C_p(K) = 0$ so $H_p(K) = 0$.

$\underline{p = 0}$ $Z_0(K) = C_0(K) = \mathbb{Z}\, a \oplus \mathbb{Z}\, b \oplus \mathbb{Z}\, c \simeq \mathbb{Z}^3$, $\quad B_0(K) = \mathbb{Z}(b-a) + \mathbb{Z}(c-b) + \mathbb{Z}(a-c)$. Note $Z_0(K) = \mathbb{Z}\, a + B_0(K)$ while (ex) $\mathbb{Z}\, a \cap B_0(K) = 0$ so isomorphism thm $\implies$

$$H_0(K) = Z_0(K)/B_0(K) \simeq \mathbb{Z}\, a/(\mathbb{Z}\, a \cap B_0(K)) = \mathbb{Z}\, a/0 \simeq \mathbb{Z}\,.$$

$\underline{p = 1}$ $B_1(K) = 0$. Let $\gamma = \gamma_1[ab] + \gamma_2[bc] + \gamma_3[ca]$. Then $\gamma \in Z_1(K)$ iff

$$0 = \partial(\gamma) = (\gamma_3 - \gamma_1)a + (\gamma_1 - \gamma_2)b + (\gamma_2 - \gamma_3)c$$

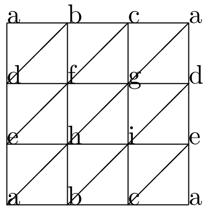
iff $\gamma_1 = \gamma_2 = \gamma_3$. Hence $Z_1(K) = \mathbb{Z}\,\gamma$ where $\gamma = [ab] + [bc] + [ca]$. We think of γ as surrounding the "hole" of the circle which lives in the homology group $H_1(K)$.

Definition 5.2 *Let K = simplicial complex. We say $\gamma, \gamma' \in Z_p(K)$ are* homologous *if $\gamma - \gamma' \in B_p(K)$, i.e. $\gamma \equiv \gamma' \mod B_p(K)$ so define the same element in $H_p(K)$.*

Let L = subcomplex of K so $C_p(L) \leq C_p(K)$. We say $\gamma \in C_p(K)$ is carried by L *if $\gamma \in C_p(L)$.*

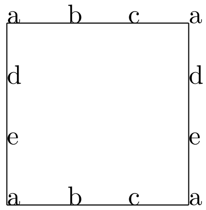
2-torus

Example 5.3 *Let T be the triangulation of the torus below*



$\underline{p=0}$ $H_0(T) \simeq \mathbb{Z}$.

Let L be the subcomplex $K^{(1)}_{abc} \cup K^{(1)}_{ade}$ so $|L|$ is a "wedge" of 2 circles.



$\underline{p=2}$

Let $\{\sigma_r\}$ be the set of all 2-simplices in T oriented anti-clockwise and Σ be their sum.

Proposition 5.4 *i. If $\gamma = \sum_r n_r\, \sigma_r \in C_2(T)$ is s.t. $\partial(\gamma)$ is carried by L, then all n_r are equal so $\gamma = n\Sigma$ for some $n \in \mathbb{Z}$.*

ii. $Z_2(T) = \mathbb{Z}\,\Sigma \simeq \mathbb{Z}$.

iii. $H_2(T) = Z_2(T)/B_2(T) = \mathbb{Z}\,\Sigma/0 \simeq \mathbb{Z}$.

Proof. (i) If σ_r, σ_s are neighbouring simplices with common internal edge ε, then the co-efficient of ε in $\partial(\gamma)$ is $\pm(n_r - n_s) = 0$. Hence all n_r are equal.

(ii) Note $\partial(\Sigma) = 0$ so $\mathbb{Z}\,\Sigma \subseteq Z_2(T)$. Also (i) $\implies$ $Z_2(T) \subseteq \mathbb{Z}\,\Sigma$.

(iii) Follows as $B_2(T) = 0$. □

$\underline{p=1}$

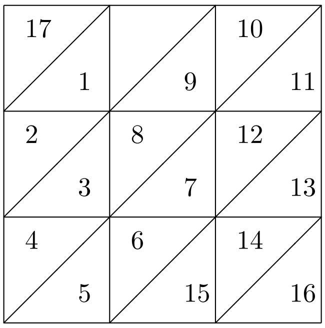
Let $\alpha = [ab] + [bc] + [ca]$, $\quad \beta = [ad] + [de] + [ea]$.

Proposition 5.5 *i. $Z_1(L) = \mathbb{Z}\,\alpha \oplus \mathbb{Z}\,\beta$.*

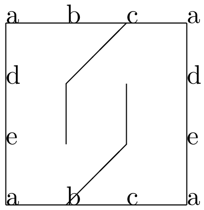
ii. Any $\gamma \in Z_1(T)$ is homologous to one carried by L, i.e. $Z_1(T) = Z_1(L) + B_1(T)$.

iii. By 5.4(i), $Z_1(L) \cap B_1(T) = 0$ so $H_1(T) = (Z_1(L) + B_1(T))/B_1(T) \simeq Z_1(L)/(Z_1(L) \cap B_1(T)) \simeq \mathbb{Z}\,\alpha \oplus \mathbb{Z}\,\beta$.

Proof. (i) follows as in e.g. 5.1 so suffice prove (ii). Adding appropriate multiple of $[bdf]$ to γ, we may replace γ with a homologous cycle with co-efficient of $[bf] = 0$. Continue inductively in order below



Can assume the γ is homologous to one carried by



But γ a cycle $\implies$ its carried by L.

$\square$

6 Chain complexes

Aim lecture Intro general algebraic framework for homology.

Chain complexes

Definition 6.1 *A* (chain) complex *of abelian groups (resp vector spaces), is a sequence*

$$C_\bullet : \ldots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \ldots$$

of abelian groups C_p (resp vector spaces) and group homomorphisms ∂_p (resp linear maps) such that $\partial^2 := \partial_{p-1}\partial_p = 0$.

Example 6.2 *If K = simplicial complex then we have the chain complex*

$$C_\bullet(K) : \ldots \longrightarrow C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \longrightarrow \ldots$$

where the ∂ are boundary operators.

Proposition-Definition 6.3 *Let $C_\bullet$ be a chain complex. The* group of p-cycles *is $Z_p(C_\bullet) = Z_p =$* $\ker \partial_p$ *and the* group of p-boundaries *is $B_p(C_\bullet) = B_p = im\ \partial_{p+1}$. Then $B_p \leq Z_p$ so we may define the* p-th homology group *to be*

$$H_p(C_\bullet) = Z_p/B_p.$$

E.g. 6.2 again. $H_p(C_\bullet(K)) = H_p(K)$.

Reduced homology

Example 6.4 *The* augmented chain complex *of a simplicial complex K.*

Let

$$\tilde{C}_p(K) = \begin{cases} C_p(K) & \text{if } p \neq -1, \\ \mathbb{Z} & \text{if } p = -1 \end{cases}$$

The boundary maps ∂ will be the same as for $C_\bullet(K)$ except now

$$\partial_0 : \tilde{C}_0(K) \longrightarrow \tilde{C}_{-1}(K) = \mathbb{Z} : a \mapsto 1$$

for any vertex $a \in K^{(0)}$.

Note $\partial_{p-1}\partial_p = 0$ holds for $p \neq 1$ since it holds in $C_\bullet(K)$, and it holds for $p = 1$ since $\partial^2(ab) = \partial(b-a) = 1 - 1 = 0$.

Definition 6.5 *We define the p*-th reduced homology of K *to be $\tilde{H}_p(K) = H_p(\tilde{C}_\bullet(K))$.*

Ex $\tilde{H}_p(K) = H_p(K)$ if $p \neq 0$ and $\tilde{H}_0(K) = \mathbb{Z}^{n-1}$ where $|K|$ has n connected components.

Chain maps

Definition 6.6 *A* subchain complex *of a chain complex $C_\bullet$, is a collection of subgroups $D_p \leq C_p$ such that $\partial_p(D_p) \subseteq D_{p-1}$. In this case, $D_\bullet$ is a chain complex too. We write $D_\bullet \leq C_\bullet$.*

E.g. If L is a subcomplex of K then $C_\bullet(L)$ is a chain subcomplex of $C_\bullet(K)$.

Let $C_\bullet, C'_\bullet$ be chain complexes. A *chain map* $f_\bullet : C_\bullet \longrightarrow C'_\bullet$ or *morphism of chain complexes* is

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{p+1} & \xrightarrow{\partial_{p+1}} & C_p & \xrightarrow{\partial_p} & C_{p-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{p+1}} & & \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle f_{p-1}} & & \\ \cdots & \longrightarrow & C'_{p+1} & \xrightarrow{\partial'_{p+1}} & C'_p & \xrightarrow{\partial'_p} & C'_{p-1} & \longrightarrow & \cdots \end{array}$$

a sequence of group homs $f_p, p \in \mathbb{Z}$ s.t. the *diagram commutes* in the sense that $\partial'_p \circ f_p = f_{p-1} \circ \partial_p$ for all p.

E.g. If L is a subcomplex of K, then the inclusion maps $C_p(L) \hookrightarrow C_p(K)$ define a chain map $\iota : C_\bullet(L) \longrightarrow C_\bullet(K)$.

Proposition 6.7 *The identity $id_\bullet : C_\bullet \longrightarrow C_\bullet$ is a chain map. Given chain maps $f_\bullet : C_\bullet \longrightarrow D_\bullet, g_\bullet : D_\bullet \longrightarrow E_\bullet$, the composite $h_\bullet := g_\bullet f_\bullet : C_\bullet \longrightarrow E_\bullet$ defined by $h_p = g_p f_p$ is a chain map.*

Proof. Good easy ex. □

N.B. All the above definitions and results hold if the chain complexes are of vector spaces and we replace group homs with linear maps.

Functoriality of H_p

Proposition-Definition 6.8 *Let $f_\bullet : C_\bullet \longrightarrow D_\bullet$ be a chain map.*

i. $f_p(Z_p(C_\bullet)) \subseteq Z_p(D_\bullet)$, $\quad f_p(B_p(C_\bullet)) \subseteq B_p(D_\bullet)$.

ii. Hence $f_\bullet$ induces a well-defined group hom

$$f_* := H_p(f_\bullet) : H_p(C_\bullet) \longrightarrow H_p(D_\bullet) : \gamma + B_p(C_\bullet) \mapsto f_p(\gamma) + B_p(D_\bullet).$$

Proof. (i) For $\gamma \in Z_p(C_\bullet)$,

$$\partial f(\gamma) = f\partial(\gamma) = f(0) = 0 \implies f(\gamma) \in Z_p(D_\bullet).$$

and

$$f(B_p(C_\bullet)) = f\partial(C_{p+1}) = \partial f(C_{p+1}) \subseteq B_p(D_\bullet).$$

Finally, (i) gives (ii). □

Proposition 6.9 *i.* $H_p(id_\bullet) = id_{H_p}$.

ii. Given chain maps $f_\bullet : C_\bullet \longrightarrow D_\bullet, g_\bullet : D_\bullet \longrightarrow E_\bullet$*, the following diagram commutes*

$$\begin{array}{ccc} H_p(C_\bullet) & \xrightarrow{(gf)_*} & H_p(E_\bullet) \\ \downarrow f_* & \nearrow g_* & \\ H_p(D_\bullet) & & \end{array}$$

i.e. $(gf)_* = g_* f_*$.

7 Homotopy

Aim lecture We recall the notion of homotopy = deformations of continuous maps from MATH3701. Most notions in algebraic topology are invariant under homotopy.

Homotopy of pairs

A *topological pair* (X, A) consists of a topological space X and a subspace A. We will identify $(X, \varnothing)$ with X. We formulate our definitions and results for pairs, and setting $A = \varnothing$ gives the definitions for top spaces.

A *continuous map of pairs* $f : (X, A) \longrightarrow (X', A')$ is a continuous map $f : X \longrightarrow X'$ such that $f(A) \subseteq A'$. Let $I = [0, 1]$. Given two continuous maps $f_0, f_1 : (X, A) \longrightarrow (X', A')$, a *homotopy* $F : f_0 \approx f_1$ is a continuous map $F : (X \times I, A \times I) \longrightarrow (X', A')$ such that $f_0(x) = F(x, 0), f_1(x) = F(x, 1)$ for all $x \in X$. We say f_0, f_1 are *homotopic* in this case.

If we further write $f_t = F(-, t) : X \longrightarrow X'$, then we can visualise the family $\{f_t | t \in [0, 1]\}$ as a continuous deformation from f_0 to f_1

DRAW PICTURE

Proposition 7.1 *The homotopic relation* $\approx$ *is an equivalence relation on the set of continuous maps* $(X, A) \longrightarrow (X', A')$.

Proof. Ex. Pictorially obvious. □

Proposition 7.2 *Let* $f_0, f_1 : (X, A) \longrightarrow (X', A'), g_0, g_1 : (X', A') \longrightarrow (X'', A'')$ *be contiuous. If* $F : f_0 \approx f_1, G : g_0 \approx g_1$ *are homotopies, then* $g_0 f_0 \approx g_1 f_1$ *i.e. composites of homotopic maps are homotopic.*

Proof. We have homotopies

$$G \circ f_0 \times \mathrm{id}_I : g_0 f_0 \approx g_1 f_0 : (X \times I, A \times I) \xrightarrow{f_0 \times \mathrm{id}_I} (X' \times I, A' \times I) \xrightarrow{G} (X'', A'').$$

$$g_1 \circ F : g_1 f_0 \approx g_1 f_1 : (X \times I, A \times I) \xrightarrow{F} (X', A') \xrightarrow{g_1} (X'', A'').$$

Transitivity $\implies g_0 f_0 \approx g_1 f_1$. □

Homotopy equivalence

Definition 7.3 *Let $f : (X, A) \longrightarrow (Y, B)$ be continuous. A* homotopy inverse *to f is a continuous map $g : (Y, B) \longrightarrow (X, A)$ such that fg is homotopic to id_Y and gf is homotopic to id_X. In this case we say f is a* homotopy equivalence *and $(X, A), (Y, B)$ are* homotopy equivalent*.*

Let A be a subspace of X & $\iota : A \hookrightarrow X$ be the inclusion. We say A is a *weak deformation retract* of X if ι is a homotopy equivalence.

Star convex

Definition 7.4 *We say that $X \subseteq \mathbb{R}^n$ is* star convex with respect to $x \in X$ *if for every point $y \in X$, the line segment xy lies in X. Note every convex set is star convex (wrt some, in fact any, point).*

E.g.

Proposition 7.5 *Let X be star convex. With notation as above, the identity map $id : X \longrightarrow X$ is homotopic to the constant map $c : X \longrightarrow \{x\} \hookrightarrow X$. In particular, $\{x\}$ is a weak deformation retract of X.*

Proof. Note that c is a left inverse to inclusion $\{x\} \longrightarrow X$ so it suffices to prove the first assertion.

Translating by $-x$, we will assume that $x = \vec{0}$. The homotopy is given by $H : X \times I \longrightarrow X : (y, t) \mapsto ty$. □

Chain homotopy

Consider two chain maps $f_\bullet, g_\bullet : C_\bullet \longrightarrow C'_\bullet$. A *chain homotopy* $s_\bullet$ between $f_\bullet$ and $g_\bullet$ is a sequence of group homs $s_p : C_p \longrightarrow C'_{p+1}$ such that

$$f_p - g_p = \partial'_{p+1} s_p + s_{p-1} \partial_p.$$

DRAW DIAGRAM

The relation with topological homotopy is suggested by the following picture:

Proposition 7.6 *Let $s_\bullet$ be a chain homotopy between $f_\bullet, g_\bullet : C_\bullet \longrightarrow C'_\bullet$.*

i. Then $H_p(f_\bullet) = H_p(g_\bullet) : H_p(C_\bullet) \longrightarrow H_p(C'_\bullet)$ for all p.

ii. In particular, if $id : C_\bullet \longrightarrow C_\bullet$ (defined by $id_p = id_{C_p}$) is homotopic to the zero map $0_p = 0 : C_\bullet \longrightarrow C_\bullet$, then $H_p(C_\bullet) = 0$.

Proof. Note (i) $\implies$ (ii), for in this case, the identity map on $H_p(C_\bullet)$ is the same as the zero map. We now prove (i).

Let $\gamma \in Z_p(C_\bullet)$. Then

$$f_p(\gamma) = g_p(\gamma) + \partial'_{p+1} s_p(\gamma) + s_{p-1}\partial_p(\gamma) = g_p(\gamma) + \partial'_{p+1} s_p(\gamma) \equiv g_p(\gamma) \mod B_p(C'_\bullet).$$

Hence $H_p(f_\bullet)(\gamma + B_p(C_\bullet)) = H_p(g_\bullet)(\gamma + B_p(C_\bullet))$. □

8 Homology of cones and spheres

Aim lecture We compute the homology of two important simplicial complexes whose polytopes are homeomorphic to the ball and sphere.

Abstract cones

Given a simplicial complex K, we pick a new vertex say $w \notin K^{(0)}$ and for definiteness, let $w > a$ for all $a \in K^{(0)}$. We define the *cone on K* to be the simplicial complex whose simplices consists of those of K as well as those of the form $a_0 \dots a_n w$ whenever $a_0 \dots a_n \in K$. (CHECK it's a simplicial complex). We denote it $K * w$.

E.g. Draw $K^{(1)}_{abc} * d$.

The following explains the terminology.

Proposition 8.1 *$|K * w|$ is star convex.*

Proof. We let $a \in K^{(0)}$ also denote the corresponding point of $|K*w|$ and show that for any $x \in |K*w|$, the line segment joining x to w lies in $|K * w|$ too. Indeed, x must lie in a geometric simplex of $|K * w|$ of form $\langle a_0 \dots a_n w \rangle$ and this is convex so closed under line segments. It contains x, w so we're done. □

Homology of cones

Proposition 8.2 *i. $\tilde{H}_p(K * w) = 0$.*

ii. $H_p(K * w) = \begin{cases} 0 & \text{if } p \neq 0, \\ \mathbb{Z} & \text{if } p = 0 \end{cases}$

Proof. Note $|K * w|$ is connected being, star convex, so (i) $\implies$ (ii). We prove (i) by constructing a chain homotopy $s_\bullet$ between $\mathrm{id}\,, 0 : \tilde{C}_\bullet(K * w) \longrightarrow \tilde{C}_\bullet(K * w)$.

We define $s_p : \tilde{C}_p(K * w) \longrightarrow \tilde{C}_{p+1}(K * w)$ by

$$s_p(\sigma) = (-1)^{p+1}\, \sigma\, w, \quad s_{p+1}(\sigma\, w) = 0$$

for σ a p-simplex in K.

Note $\partial(\sigma\, w) = (\partial\, \sigma)w + (-1)^{p+1}\, \sigma$. Hence

$$(\partial s + s\partial)\, \sigma = (-1)^{p+1}((\partial\, \sigma)w + (-1)^{p+1}\, \sigma) + (-1)^p(\partial\, \sigma)w = \sigma\, .$$

Similarly,

$$(\partial s + s\partial)\, \sigma\, w = s((\partial\, \sigma)w + (-1)^{p+1}\, \sigma) = \sigma\, w.$$

Hence $\partial s + s\partial = \mathrm{id}$ and the proposition is proved.

□

Homology of spheres

Definition 8.3 *A simplicial complex K is said to be* acyclic *if $\tilde{H}_p(K) = 0$ for all p.*

Theorem 8.4 *Consider the n-simplex $\sigma = a_0 \dots a_n$.*

i. K_σ is acyclic.

ii. Let $\Sigma^{n-1} = K_\sigma^{(n-1)}$ where $n > 1$. Then

$$H_p(\Sigma^{n-1}) = \begin{cases} 0 & \text{if } p \neq 0 \text{ or } n-1 \\ \mathbb{Z} & \text{if } p = 0 \text{ or } n-1 \end{cases}.$$

Furthermore, $Z_{n-1}(\Sigma^{n-1})$ is generated by $\partial\, \sigma$.

Proof. (i) Note that K_σ is the cone $K_{a_0 \dots a_{n-1}} * a_n$ so it has zero reduced homology by 8.2.

(ii) Note that $H_0(\Sigma^{n-1}) = \mathbb{Z}$ as $|\Sigma^{n-1}| \simeq S^{n-1}$ which is connected when $n > 1$. We compute $H_p(\Sigma^{n-1})$ for $p > 0$. Now $\tilde{C}_p(K_\sigma) = \tilde{C}_p(\Sigma^{n-1})$ for $p \neq n$ so $0 = \tilde{H}_p(K_\sigma) = \tilde{H}_p(\Sigma^{n-1})$ except possibly when $p = n-1, n, n+1$.

Now Σ^{n-1} is a subcomplex of K_σ so there is a chain map

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C_{n-1}(\Sigma^{n-1}) & \longrightarrow & \cdots \\ & & \downarrow & & \| & & \| \\ 0 & \longrightarrow & C_n(K_\sigma) = \mathbb{Z}\,\sigma & \longrightarrow & C_{n-1}(K_\sigma) & \longrightarrow & \cdots \end{array}.$$

Note $0 = H_n(\Sigma^{n-1}) = H_{n+1}(\Sigma^{n-1})$. Also,

$$H_{n-1}(\Sigma^{n-1}) = Z_{n-1}(\Sigma^{n-1}) = Z_{n-1}(K_\sigma) \overset{(i)}{=} B_{n-1}(K_\sigma) = \mathbb{Z}(\partial\, \sigma).$$

□

Suspension

Let K be a simplicial complex. We consider two new vertices $w, w' \notin K^{(0)}$ and form the new simplicial complex $S(K) = K * w \cup K * w'$ called the *suspension of* K. This is an extremely important operation in algebraic topology although we won't get much opportunity to look at it.

E.g. $|S(K^{(1)}_{abc})|$.

Given any $\sigma \in K$ a p-simplex, we have a homomorphism

$$\nu : \tilde{C}_p(K) \longrightarrow \tilde{C}_{p+1}(S(K)) : \sigma \mapsto \sigma\, w - \sigma\, w'.$$

Needless to say, this cannot give a chain map $\tilde{C}_\bullet(K) \longrightarrow \tilde{C}_\bullet(S(K))$, but if we re-index $\tilde{C}_\bullet(S(K))$ by letting $\tilde{C}_\bullet(S(K))[1]$ be the same sequence of abelian groups and boundary maps but indexed so $\tilde{C}_p(S(K))[1] = \tilde{C}_{p+1}(S(K))$, then ν is a chain map. Indeed,

$$\partial\nu\, \sigma = \partial(\sigma\, w - \sigma\, w') = \partial\, \sigma\, w + (-1)^{p+1}\, \sigma - \partial\, \sigma\, w' - (-1)^{p+1}\, \sigma = \nu\partial\, \sigma\,.$$

Thus by functoriality proposition 6.8, we obtain a homomophism $\tilde{H}_p(K) \longrightarrow \tilde{H}_{p+1}(S(K))$.

It turns out this is actually an isomorphism, a fact we won't prove as it is usually proved using the Mayer-Vietoris sequence. We will however, see a similar result for singular homology. You might like to check the isomorphism on a toy example, see problem sheets.

9 Homology with co-efficients. Betti numbers

Aim lecture Introduce numerical invariants called Betti numbers derived from homology with co-efficients.

Today, $\mathbb{F}$ denotes a field and K = simplicial complex.

Homology with Co-efficients

We define a chain complex $C_\bullet(K, \mathbb{F})$ of vector spaces over $\mathbb{F}$ by:

$$C_p(K, \mathbb{F}) = \bigoplus_{\sigma \in K \text{ is a } p \text{ simplex}} \mathbb{F}\, \sigma$$

that is, the vector space with basis the p-simplices of K. The boundary map ∂ is the unique $\mathbb{F}$-linear map such that

$$\partial(a_0 \dots a_p) = \sum_{i=0}^{p} (-1)^i a_0 \dots \hat{a}_i \dots a_p.$$

Our old proof also gives

Proposition-Definition 9.1 *$C_\bullet(K, \mathbb{F})$ is a chain complex of $\mathbb{F}$-spaces. Its homology, denoted $H_p(K, \mathbb{F})$ is called the* homology of K with co-efficients in $\mathbb{F}$. *These are $\mathbb{F}$-spaces.*

The following examples can be proved using the same techniques as for usual homology.

E.g.

- Given a cone $K * w$ we have $H_p(K * w, \mathbb{F}) = 0$ for $p > 0$ and $H_0(K * w, \mathbb{F}) = \mathbb{F}$.
- Consider the "boundary" of an n-simplex σ, denote it $\Sigma^{n-1} = K^{(n-1)}_\sigma$. Then

$$H_p(\Sigma^{n-1}, \mathbb{F}) = \begin{cases} 0 & \text{if } p \neq 0 \text{ or } n-1 \\ \mathbb{F} & \text{if } p = 0 \text{ or } n-1 \end{cases}.$$

- Let T be the triangulation of the 2-torus $\mathbb{T}^2$ given in e.g. 3.7. Then $H_0(T, \mathbb{F}) = H_2(T, \mathbb{F}) = \mathbb{F}, H_1(T, \mathbb{F}) = \mathbb{F}^2$ and all other homology is 0.

Unfortunately, we will not prove the following result whose proof depends on the universal co-efficient theorem.

Theorem 9.2 *If $H_p(K) \simeq \mathbb{Z}^r \oplus T$ for some finite abelian group T, then $H_p(K, \mathbb{F}) \simeq \mathbb{F}^r$ for any field $\mathbb{F}$ of characteristic 0.*

Definition 9.3 *The* p-th Betti number of K *is the vector space dimension* $\beta_p(K) := \dim_{\mathbb{Q}} H_p(K, \mathbb{Q})$.

Euler number

Definition 9.4 *The* Euler number of K *is the alternating sum of the Betti numbers:*

$$e(K) := \sum_{p=0}^{\infty} (-1)^p \dim_{\mathbb{Q}} H_p(K, \mathbb{Q}).$$

N.B. The sum is finite.

Remarkably, we will show that the Euler number can be computed without calculating homology!

Lemma 9.5 *Let $W < V$ be finite dim $\mathbb{F}$-spaces. Then* $\dim V/W = \dim V - \dim W$.

Proof. Let W' be a vector space complement to W in V. Then $W + W' = V, W \cap W' = 0$ so by the isomorphism theorem (for vector spaces),

$$V/W = (W + W')/W \simeq W'.$$

Hence $\dim V/W = \dim W' = \dim V - \dim W$. □

Theorem 9.6

$$e(K) = \sum_{p=0}^{\infty} (-1)^p \dim_{\mathbb{Q}} C_p(K, \mathbb{Q}).$$

Proof. The first isomorphism thm shows that $B_p(K, \mathbb{Q}) \simeq C_{p+1}(K, \mathbb{Q})/Z_{p+1}(K, \mathbb{Q})$ so from lemma 9.5 we have

$$\dim B_p(K, \mathbb{Q}) = \dim C_{p+1}(K, \mathbb{Q}) - \dim Z_{p+1}(K, \mathbb{Q}).$$

Lemma 9.5 also shows that

$$\begin{aligned}
e(K) &= \sum_p (-1)^p \dim_{\mathbb{Q}} H_p(K, \mathbb{Q}) \\
&= \sum_p (-1)^p \dim_{\mathbb{Q}} Z_p(K, \mathbb{Q}) - \sum_p (-1)^p \dim_{\mathbb{Q}} B_p(K, \mathbb{Q}) \\
&= \sum_p (-1)^p \dim_{\mathbb{Q}} Z_p(K, \mathbb{Q}) - \sum_p (-1)^p \dim_{\mathbb{Q}} C_{p+1}(K, \mathbb{Q}) + \sum_p (-1)^p \dim_{\mathbb{Q}} Z_{p+1}(K, \mathbb{Q}) \\
&= \sum_p (-1)^{p+1} \dim_{\mathbb{Q}} C_{p+1}(K, \mathbb{Q})
\end{aligned}$$

□

Examples

Example 9.7 *Triangulated surfaces*

If K is a simplicial complex, all of whose simplices have dim ≤ 2, then

$$\dim C_0(K, \mathbb{F}) = \text{no. vertices of } K$$
$$\dim C_1(K, \mathbb{F}) = \text{no. edges of } K$$
$$\dim C_2(K, \mathbb{F}) = \text{no. faces of } K$$

so the thm says our new definition agrees with the old.

Let T be the triangulation of the 2-torus in the previous e.g. By definition, $e(T) = 1 - 2 + 1 = 0$.

Example 9.8 *Triangulations of S^1.*

Let K_n be the n-gon $K_n = K_{a_1a_2} \cup K_{a_2a_3} \cup \ldots \cup K_{a_{n-1}a_n} \cup K_{a_1a_n}$. Hence $|K_n| \simeq S^1$.

PICTURE

Note $H_0(K_n, \mathbb{Q}) = \mathbb{Z}$ whilst $H_1(K_n, \mathbb{Q}) = \mathbb{Q}([a_1a_2] + \ldots [a_na_1])$. Hence $e(K_n) = 1 - 1 = 0$. Alternatively, note

$$\dim C_0(K, \mathbb{Q}) - \dim C_1(K, \mathbb{Q}) = n - n = 0.$$

Example 9.9 *Let Σ^{n-1} be the boundary of the n-simplex as in the first example.*

Here $e(\Sigma^{n-1}) = 1 + (-1)^{n-1}$.

10 Categories and functors

Aim lecture Intro higher order abstraction notions of categories and functors. This is the language which allows us to use algebra to study topology.

Categories

Definition A *category* $\mathcal{C}$ consists of the following data:

i. a class of *objects* denoted $\operatorname{Obj}\mathcal{C}$,

ii. disjoint sets of *morphisms* $\operatorname{Hom}_\mathcal{C}(X, Y)$ for any $X, Y \in \operatorname{Obj}\mathcal{C}$,

iii. for any $X, Y, Z \in \operatorname{Obj}\mathcal{C}$, a *composition map*

$$\operatorname{Hom}_\mathcal{C}(Y, Z) \times \operatorname{Hom}_\mathcal{C}(X, Y) \longrightarrow \operatorname{Hom}_\mathcal{C}(X, Z) : (f, g) \mapsto fg$$

satisfying the following axioms

i. composition is associative whenever it's defined

ii. for each $X \in \operatorname{Obj}\mathcal{C}$, there exists an element $\operatorname{id}_X \in \operatorname{Hom}_\mathcal{C}(X, X)$ such that

$$\operatorname{id}_X g = g, \quad f \operatorname{id}_X = f$$

whenever the LHS is defined. We call id_X the *identity on X*.

When lazy we write $X \in \mathcal{C}$ instead of $X \in \mathrm{Obj}\,\mathcal{C}$. For $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ we also write $f : X \longrightarrow Y$.

Example 10.1 *Below are some categories. Composition is the usual composition of functions unless otherwise stated. The same goes for the identity.*

- $\mathcal{C} = \underline{\mathrm{Grp}}$, the category of groups. $\mathrm{Obj}\,\mathcal{C}$ = class of groups, $\mathrm{Hom}_{\underline{\mathrm{Grp}}}(X,Y)$ = the set of group homomorphisms of form $\phi : X \longrightarrow Y$.
- $\mathcal{C} = \underline{\mathrm{Ab}}$. $\mathrm{Obj}\,\mathcal{C}$ = class of abelian groups, $\mathrm{Hom}_{\underline{\mathrm{Ab}}}(X,Y)$ = the set of group homomorphisms of form $\phi : X \longrightarrow Y$.
- $\mathcal{C} = \underline{\mathrm{Top}}$. $\mathrm{Obj}\,\mathcal{C}$ = class of topological spaces, $\mathrm{Hom}_{\underline{\mathrm{Top}}}(X,Y)$ = the set of continuous maps of form $\phi : X \longrightarrow Y$.
- $\mathcal{C} = \underline{\mathrm{TopPair}}$. $\mathrm{Obj}\,\mathcal{C}$ = class of topological pairs (X,A), $\mathrm{Hom}_{\underline{\mathrm{TopPair}}}((X,A),(Y,B))$ = the set of continuous maps of pairs.
- $\mathcal{C} = \underline{\mathrm{Ch}}$. $\mathrm{Obj}\,\mathcal{C}$ = class of chain complexes, $\mathrm{Hom}_{\underline{\mathrm{Ch}}}$ = chain maps. Composition is the composite of chain maps (OK by 6.7).
- $\mathcal{C} = \underline{\mathrm{HTop}}$. $\mathrm{Obj}\,\mathcal{C}$ = class of topological spaces, $\mathrm{Hom}_{\underline{\mathrm{HTop}}}(X,Y)$ = the set of homotopy equivalences classes $[f]$ of continuous maps $f : X \longrightarrow Y$. Composite $[f][g] := [fg]$ well-defined by proposition 7.2. We call this the *homotopy category.*

Isomorphisms

Definition 10.2 *Let $\mathcal{C}$ be a category and $f \in Hom_{\mathcal{C}}(X,Y)$. We say f is an* isomorphism in $\mathcal{C}$ *if there is some morphism $g \in Hom_{\mathcal{C}}(Y,X)$ such that $fg = id_Y$ and $gf = id_X$. In this case we say g is the ($\because$ unique)* inverse *of f and X, Y are* isomorphic *in $\mathcal{C}$.*

Example 10.3 *We give the notion of isomorphism for various categories $\mathcal{C}$.*

- $\mathcal{C} = \underline{\mathrm{Grp}}$. Isomorphisms are group isomorphisms.
- $\mathcal{C} = \underline{\mathrm{Top}}$. Isomorphisms are homeomorphisms.
- $\mathcal{C} = \underline{\mathrm{HTop}}$. Isomorphisms are called *homotopy equivalences* by definition.

Proposition 10.4 *The composite of isomorphisms is an isomorphism.*

Proof. Ex. □

Functors

Let $\mathcal{C}, \mathcal{D}$ be categories.

Definition 10.5 *A (covariant) functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of the data:*

i. A function $Obj\mathcal{C} \longrightarrow Obj\mathcal{D} : X \mapsto F(X)$ and,

ii. for each $X, Y \in \mathcal{C}$ a function

$$F : Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$$

such that

i. $F(id_X) = id_{F(X)}$ *and,*

ii. $F(fg) = F(f)F(g) : F(X) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(Z)$ *given* $X \xrightarrow{g} Y \xrightarrow{f} Z$ *in* $\mathcal{C}$*.*

We also say in this case, $F(X)$ *is* functorial *in* X*.*

Proposition 6.9 can be restated as

Proposition 10.6 $H_p : \underline{Ch} \longrightarrow \underline{Ab}$ *is a covariant functor.*

Example 10.7 *We have a functor* $Q : \underline{Top} \longrightarrow \underline{HTop}$ *defined by* $Q(X) = X$ *and* $Q(f) = [f]$*.*

Proposition 10.8 *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be a functor and* $f \in Hom_{\mathcal{C}}$ *be an isomorphism in* $\mathcal{C}$*. Then* $F(f)$ *is also an isomorphism in* $\mathcal{D}$*.*

Proof. Let $g \in \mathrm{Hom}_{\mathcal{C}}$ be the inverse to f in $\mathcal{C}$. Then $F(g)$ is the inverse to $F(f)$ for

$$F(f)F(g) = F(fg) = F(\mathrm{id}) = \mathrm{id}$$

and similarly $F(g)F(f) = \mathrm{id}$. □

We mostly use this in the contrapositive form

Corollary 10.9 *Let* $F : \underline{HTop} \longrightarrow \underline{Ab}$ *be a functor and* $X, Y \in \underline{Top}$*. If* $F(X) \not\cong F(Y)$ *then* X, Y *are not homotopically equivalent and in particular, not homeomorphic.*

11 Simplicial maps. Homotopy groups

Aim lecture We clarify the relationship between simplicial complexes, topological spaces, chain complexes and groups by using functors.

Category of simplicial complexes

$$K, L = \text{simplicial complexes}$$

Below, we let $a \in K^{(0)}$ also denote the corresponding point of $|K|$.

Proposition-Definition 11.1 *A* simplicial map $f : K \longrightarrow L$ *is a map on vertices* $f : K^{(0)} \longrightarrow L^{(0)}$ *such that if* $a_0 \dots a_p \in K$ *then* $f(a_0) \dots f(a_p)$ *are the (not necessarily distinct) vertices of a simplex in* L*.*

Given such a simplicial map, the affine linear maps $l(\langle a_0 \dots a_p \rangle, \langle f(a_0) \dots f(a_p) \rangle)$ *glue together to form a continuous map denoted* $|f| : |K| \longrightarrow |L|$*.*

Proof. This follows immediately from the gluing lemma 2.6. □

E.g.

Proposition 11.2 *There is a category* $\mathcal{C} = \underline{Simp}$ *of simplicial complexes with* $Obj\,\mathcal{C}$ *= class of simplicial complexes,* $Hom_{\mathcal{C}}$ *= simplicial maps and composition the usual composition of (vertex) maps.*

Proof. It suffices to show that the composite of simplicial maps $K \xrightarrow{f} L \xrightarrow{g} M$ is simplicial for composition is associative the identity is clearly simplicial. This is clear. □

Polytope functor

Proposition-Definition 11.3 *We have a* polytope functor $|\cdot| : \underline{Simp} \longrightarrow \underline{Top}$ *defined on objects by* $K \mapsto |K|$ *and morphisms by* $f \mapsto |f|$.

Proof. Note $l(\langle \sigma \rangle, \langle \sigma \rangle) = \text{id}$ so $|\text{id}_K| = \text{id}_{|K|}$. Consider simplicial maps $K \xrightarrow{f} L \xrightarrow{g} M$. Note $|gf| = |g||f|$ since they both send

$$\sum_i t_i a_i \mapsto \sum_i t_i f(a_i) \mapsto \sum_i t_i (gf)(a_i).$$

□

Proposition 11.4 *Given functors* $F : \mathcal{C} \longrightarrow \mathcal{D}, G : \mathcal{D} \longrightarrow \mathcal{E}$*, there is a* composite *functor* $GF : \mathcal{C} \longrightarrow \mathcal{E}$ *defined on objects by* $GF(X) = G(F(X))$ *and morphisms by* $GF(f) = G(F(f))$.

Proof. Good ex. □

We often call the composed functor $Q|\cdot| : \underline{\text{Simp}} \longrightarrow \underline{\text{Top}} \longrightarrow \underline{\text{HTop}}$ the polytope funtor as well and, abusing notation, denote it also by $|\cdot|$.

Homology functor

We have a functor $C_\bullet : \underline{\text{Simp}} \longrightarrow \underline{\text{Ch}}$ defined as follows. On objects it maps $K \mapsto C_\bullet(K)$. On a simplicial map $f : K \longrightarrow L$ the chain map $c_\bullet := C_\bullet(f) : C_\bullet(K) \longrightarrow C_\bullet(L)$ is defined by

$$c_p : C_p(K) \longrightarrow C_p(L) : a_0 \dots a_p \mapsto \begin{cases} [f(a_0) \dots f(a_p)] & \text{if } f(a_0), \dots, f(a_p) \text{ are distinct,} \\ 0 & \text{otherwise} \end{cases}.$$

Note $c_\bullet$ is a chain map since if $f(a_0), \dots, f(a_p)$ are distinct,

$$\partial c_p(a_0 \dots a_p) = \sum_i (-1)^i [f(a_0) \dots \widehat{f(a_i)} \dots f(a_p)] = c_p(\partial(a_0 \dots a_p)).$$

If they are not distinct, one can show $c_p(\partial(a_0 \dots a_p)) = 0$ with a little care taking cases ex.

It's easy to see $C_\bullet$ is functorial ex. We thus obtain a *p-th homology functor* $H_p : \underline{\text{Simp}} \longrightarrow \underline{\text{Ab}}$ as the composite

$$\underline{\text{Simp}} \xrightarrow{C_\bullet} \underline{\text{Ch}} \xrightarrow{H_p} \underline{\text{Ab}}.$$

Similarly, we can define *homology with co-efficients* in a field $\mathbb{F}$ functor $H_p(-, \mathbb{F}) : \underline{\text{Simp}} \longrightarrow \underline{\text{Vect}_\mathbb{F}}$ where $\underline{\text{Vect}_\mathbb{F}}$ is the category of $\mathbb{F}$-spaces and $\mathbb{F}$-linear maps.

Homotopy groups

Consider the *category of sets* $\underline{\text{Set}}$ whose objects are sets, and morphisms are functions.

Proposition 11.5 *Let* $\mathcal{C}$ *be a category and* $C \in \mathcal{C}$. *We have a functor* $Hom_\mathcal{C}(C, -) : \mathcal{C} \longrightarrow \underline{Set}$ *defined on an object* $X \in \mathcal{C}$ *as the set of morphisms* $Hom_\mathcal{C}(C, X)$ *and given a morphism* $f : X \longrightarrow Y$ *in* $\mathcal{C}$,

$$Hom_\mathcal{C}(C, f) : Hom_\mathcal{C}(C, X) \longrightarrow Hom_\mathcal{C}(C, Y) : g \mapsto fg.$$

Proof. Note $\mathrm{Hom}_{\mathcal{C}}(C, \mathrm{id}_X) : g \mapsto \mathrm{id}\, g = g$ is the identity on $\mathrm{Hom}_{\mathcal{C}}(C, X)$. Also given $X \xrightarrow{f} Y \xrightarrow{f'} Z$ in $\mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{C}}(C, f'f)(g) = f'fg = \mathrm{Hom}_{\mathcal{C}}(C, f')(fg) = \mathrm{Hom}_{\mathcal{C}}(C, f')\,\mathrm{Hom}_{\mathcal{C}}(C, f)(g)$$

and $\mathrm{Hom}_{\mathcal{C}}(C, -)$ is a functor. □

Let $\underline{\mathrm{HPtTop}}$ be the *homotopy category of pointed topological spaces* whose objects are topological pairs of the form (X, x) for some $x \in X$, and morphisms are homotopy equivalence classes of continuous maps of pairs. We define the functor $\pi_n : \underline{\mathrm{HPtTop}} \longrightarrow \underline{\mathrm{Set}}$ to be $\pi_n = \mathrm{Hom}_{\underline{\mathrm{HPtTop}}}((S^n, \mathrm{pt}), -)$.

Theorem 11.6 *For each $(X, x) \in \underline{HPtTop}$, there is a group structure on $\pi_n(X, x)$ which makes π_n a functor $\underline{HPtTop} \longrightarrow \underline{Grp}$ (i.e. $\pi_n(f)$ is also a group homomorphism). We call $\pi_n(X, x)$ the* n-th homotopy group *of (X, x).*

Proof. We will not prove this, but you should have seen π_1 in MATH3701. □

12 Homology: the topological invariant

Aim lecture We state the Main Theorem of this course, the existence of a homology functor $H_* : \underline{\mathrm{HTop}} \longrightarrow \underline{\mathrm{Ab}}$ and its computability via homology of simplicial complexes.

Let $X =$ topological space

Natural isomorphisms

It's rare that two functors found in different situations are equal, usually they are only *naturally isomorphic* in the sense below.

Definition 12.1 *Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors. A* natural transformation *$\eta : F \xrightarrow{\sim} G$ consists of morphisms (in $\mathcal{D}$) $\eta_X : F(X) \longrightarrow G(X)$ for each $X \in \mathcal{C}$ such that for every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ the square below is commutative*

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

If all η_X's are isomorphisms, we say η is a natural isomorphism, *and F, G are* naturally isomorphic. *We sometimes write $F \simeq G$.*

E.g. Let $\underline{\mathrm{Vect}}^3_{\mathbb{F}}$ be the category of triples (V_1, V_2, V_3) of $\mathbb{F}$-spaces and morphisms are triples $(f_1, f_2, f_3) : (V_1, V_2, V_3) \longrightarrow (V_1', V_2', V_3')$ of linear maps $f_i : V_i \longrightarrow V_i'$. Note that we can form the direct sum of two vector spaces and of linear maps so we obtain two functors:

$$- \oplus (- \oplus -) : \underline{\mathrm{Vect}}^3_{\mathbb{F}} \longrightarrow \underline{\mathrm{Vect}}_{\mathbb{F}} : (V_1, V_2, V_3) \mapsto V_1 \oplus (V_2 \oplus V_3)$$
$$(- \oplus -) \oplus - : \underline{\mathrm{Vect}}^3_{\mathbb{F}} \longrightarrow \underline{\mathrm{Vect}}_{\mathbb{F}} : (V_1, V_2, V_3) \mapsto (V_1 \oplus V_2) \oplus V_3$$

On morphisms, $- \oplus (- \oplus -)$ sends (f_1, f_2, f_3) to $f_1 \oplus (f_2 \oplus f_3)$ and similarly for $(- \oplus -) \oplus -$. The two functors are not equal, but they are naturally isomorphic via

$$\eta_{(V_1, V_2, V_3)} : V_1 \oplus (V_2 \oplus V_3) \simeq (V_1 \oplus V_2) \oplus V_3 : (v_1, (v_2, v_3)) \mapsto ((v_1, v_2), v_3).$$

Ex. check naturality.

Main Theorem

Theorem 12.2 *For each $p \in \mathbb{N}$, there is a functor $H_p : \underline{HTop} \longrightarrow \underline{Ab}$ (resp $\tilde{H}_p, H_p(-,\mathbb{F}) : \underline{HTop} \longrightarrow \underline{Vect_{\mathbb{F}}}$) such that the simplicial homology functor $H_p : \underline{Simp} \longrightarrow \underline{Ab}$ (resp reduced homology, homology with co-efficients) is naturally isomorphic to the composite functor*

$$\underline{Simp} \xrightarrow{|\cdot|} \underline{HTop} \xrightarrow{H_p} \underline{Ab}$$

(resp ...).

In particular, given any $K \in \underline{Simp}$, we have $H_p(|K|) \simeq H_p(K)$.

Proof. much later in section 23 □

We call $H_p(X)$ the *p-th homology group of* X, and similarly for $\tilde{H}_p(X), H_p(X,\mathbb{F})$.

Homology of some topological spaces

Since functors preserve isomorphisms, homotopically equivalent spaces have isomorphic homology groups.

i. Suppose X is homotopy equivalent to a point, e.g. this occurs if X is star convex by proposition 7.5. Then $\tilde{H}_p(X) \simeq \tilde{H}_p(\mathrm{pt})$ is isomorphic to the reduced homology of the 0-simplex K_a i.e. 0.

ii. If σ is an n-simplex then $H_p(S^{n-1}) \simeq H_p(K_\sigma^{(n-1)}) = \mathbb{Z}$ if $p = 0$ or $n-1$ and 0 otherwise.

iii. From e.g. 5.3 we know

$$H_p(\mathbb{T}^2) \simeq \begin{cases} \mathbb{Z} & \text{if } p = 0,2 \\ \mathbb{Z}^2 & \text{if } p = 1 \\ 0 & \text{otherwise} \end{cases}$$

Topological applications

Proposition 12.3 *If $m \neq n$, then S^n, S^m are not homeomorphic.*

Proof. Since $H_m(S^m) \simeq \mathbb{Z} \not\simeq 0 = H_m(S^n)$. □

Similarly, we also see that the 2-torus and 2-sphere are not homeomorphic.

Lemma 12.4 *The space $U = \mathbb{R}^n - 0$ is homotopy equivalent to S^{n-1}.*

Proof. We show the retraction $f : U \longrightarrow S^{n-1} : v \mapsto v/|v|$ is inverse to the inclusion $\iota : S^n \hookrightarrow U$ in $\underline{\text{HTop}}$. Since $f\iota = \mathrm{id}_{S^{n-1}}$, it suffices to show that ιf is homotopy equivalent to id_U.

The desired homotopy is given by $H : U \times I \longrightarrow U : (v,t) \mapsto (1-t)v + \frac{tv}{|v|}$. □

Theorem 12.5 *If $m \neq n$, then $\mathbb{R}^m, \mathbb{R}^n$ are not homeomorphic.*

Proof. Suppose by contradiction that $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is an homeomorphism. Then $\mathbb{R}^m - 0$ is homeomorphic to $\mathbb{R}^n - f(0)$. The lemma 12.4 shows then that S^{m-1} and S^{n-1} are homotopically equivalent (since homotopy equivalence is an equivalence relation ex). This is false as they have different homology groups in degree $m-1$ and $n-1$. □

Betti numbers of topological spaces

We define the *p-th Betti number of X* to be $\dim_{\mathbb{Q}} H_p(X, \mathbb{Q})$. If all these numbers are finite and $H_p(X) = 0$ for $p \gg 0$, then we can define the *Euler number of X* to be

$$e(X) = \sum_p (-1)^p \dim_{\mathbb{Q}} H_p(X, \mathbb{Q}).$$

If X and Y are homotopically equivalent then the vector spaces $H_p(X, \mathbb{Q}) \simeq H_p(Y, \mathbb{Q})$ so they have the same Betti numbers, and Euler number (if defined).

Proposition 12.6 *Homotopically equivalent topological spaces have the same Betti numbers, and hence the same Euler number, if it's defined.*

Remark Recall that if X has a triangulation $h : |K| \longrightarrow X$, then it has infinitely many triangulations. From our main theorem and theorem 9.6,

$$e(X) = \sum_p (-1)^p \dim_Q C_p(K, \mathbb{Q}).$$

The remarkable fact is that though the individual dimensions $\dim_Q C_p(K, \mathbb{Q})$ vary as you vary K, the alternating sum does not.

13 Topological applications of the degree map

Aim lecture Intro the degree of a self map of spheres, and give applications to Brouwer's fixed point theorem and the fundamental thm of algebra.

Degree

Need

Lemma 13.1 *Any group hom $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}$ is multiplication by d where $d = \phi(1)$.*

Proof. $\phi(n) = n\phi(1) = dn.$ □

Let $f : S^n \longrightarrow S^n$ be cont. The lemma shows that $H_p(f) : H_n(S^n) \simeq \mathbb{Z} \longrightarrow H_n(S^n)$ must be multiplication by an integer $d =: \deg f$ which we call the *degree* of f.

Here are some basic properties.

Proposition 13.2 *i. If $f \approx g$ (i.e homotopy equiv), then* $\deg f = \deg g$.

ii. $\deg id = 1$.

iii. If f extends to a continuous map $h : B^{n+1} \longrightarrow S^n$, then $\deg f = 0$.

iv. $\deg fg = (\deg f)(\deg g)$.

Proof. (i) $[f] = [g] \implies H_n([f]) = H_n([g])$.
(ii) H_n a functor $\implies H_n(\mathrm{id}) = \mathrm{id}$ i.e. multn by 1.
(iii) $H_n(f)$ factors as

$$H_n(S^n) \xrightarrow{H_n(\mathrm{inc})} H_n(B^{n+1}) = 0 \xrightarrow{H_n(h)} H_n(S^n)$$

so is 0.
(iv) $H_n(fg) = H_n(f)H_n(g)$ is the composite of multiplications

$$H_n(S^n) \xrightarrow{\deg g} H_n(S^n) \xrightarrow{\deg f} H_n(S^n)$$

which is multiplication by $(\deg f)(\deg g)$. □

Degree and winding
Below, we view $S^1 = \{z \in \mathbb{C} | |z| = 1\}$. Recall, that $\mathbb{Z}$ has 2 generators only, ± 1.

Proposition 13.3 *Let* $f : S^1 \longrightarrow S^1 : z \mapsto z^d$ *(DRAW PICTURE). Then* $\deg f = d$.

Proof. Best understood pictorially (see picture). We'll only prove $\deg f = \pm d$ which is all we need.

We use 2 different triangulations of S^1. Consider vertex sets $K_1^{(0)} = \mathbb{Z}/3d\mathbb{Z}, K_2^{(0)} = \mathbb{Z}/3\mathbb{Z}$ and simplicial complexes

$$K_1 = \cup_{a\in\mathbb{Z}/3d\mathbb{Z}} K_{a,a+1}, \quad K_2 = \cup_{a\in\mathbb{Z}/3\mathbb{Z}} K_{a,a+1}.$$

Hence $|K_1|$ is a regular $3d$-gon and $|K_2|$ is an equilateral △. Radial projections $h_1 : |K_1| \longrightarrow S^1, h_2 : |K_2| \longrightarrow S^1$ give triangulations of the circle.
Consider the simplicial map

$$\phi : K_1^{(0)} = \mathbb{Z}/3d\mathbb{Z} \longrightarrow K_2^{(0)} = \mathbb{Z}/3\mathbb{Z} : a + 3d\mathbb{Z} \mapsto a + 3\mathbb{Z}.$$

Observe that $f \approx h_2 \circ |\phi| \circ h_1^{-1}$ (ex in problem set 2) and that ϕ_* sends the generator $\sum_{a\in\mathbb{Z}/3d\mathbb{Z}}[a\ a+1] \mapsto d\sum_{a\in\mathbb{Z}/3\mathbb{Z}}[a\ a+1]$, d times the generator $\sum_{a\in\mathbb{Z}/3\mathbb{Z}}[aa+1]$ for $H_1(K_2)$. Now $H_1(f)$ is the composite of the top row in the commutative diagram below.

$$\begin{array}{ccccccc} H_1(S^1) & \xrightarrow{H_1(h_1^{-1})} & H_1(|K_1|) & \xrightarrow{H_1(|\phi|)} & H_1(|K_2|) & \xrightarrow{H_1(h_2)} & H_1(S^1) \\ & & \downarrow & & \downarrow & & \\ & & H_1(K_1) & \xrightarrow{H_1(\phi)} & H_1(K_2) & & \end{array}$$

where the vertical arrows are the natural isomorphisms between simplicial and topological homology guaranteed by theorem 12.2. We see then that the top row must be multn by $\pm d$. □

Brouwer fixed point theorem

Theorem 13.4 *Any cont map* $h : B^{n+1} \longrightarrow B^{n+1}$ *has a fixed point.*

Proof. If h has no fixed points, we have a cont map $h' : B^{n+1} \longrightarrow S^n$ defined by

$$h'(v) = \frac{v - h(v)}{|v - h(v)|}.$$

We consider the restriction $f := h'|_{S^n} : S^n \longrightarrow S^n$ which has degree 0 by lemma 13.2(iii). However, f is homotopic to id by the homotopy

$$H(v,t) = \frac{v - th(v)}{|v - th(v)|}$$

(note denominator non-zero), so $\deg f = 1$ too, a contradiction. □

Fundamental theorem of algebra

Theorem 13.5 *let $p(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_0$ be a complex polynomial of degree $d > 0$. Then p has a complex root in the disc $B_c := \{z \in \mathbb{C} \| z| \leq c\}$ for any $c > d \max\{1, |a_{d-1}|, \ldots, |a_0|\}$.*

Proof. We view p as a continuous map $\mathbb{C} \longrightarrow \mathbb{C}$. Suppose the theorem is false so p restricts to a map $p|_{B_c} : B_c \longrightarrow \mathbb{C} - 0 =: U$. Let S_c = boundary of B_c and $r : U \longrightarrow S_c : z \mapsto \frac{cz}{|z|}$ be a retraction. We consider the composite $f = rp|_{S_c} : S_c \longrightarrow S_c$. First note that f extends to B_c so $\deg f = 0$ by proposition 13.2.

Let $q : S_c \longrightarrow U : z \mapsto z^d$. We wish to show that f is homotopy equivalent to $f' : z \mapsto c^{-(d-1)}z^d = rq(z)$. This will give the desired contradiction as proposition 13.3 shows $\deg f' = d$.

It suffices by Proposition 7.2 to show that $p|_{S_c}, q : S_c \longrightarrow U$ are homotopy equivalent. We show that $H : S_c \times I \longrightarrow U$ defined below is a homotopy.

$$H(z,t) = z^d + t(a_{d-1}z^{d-1} + \ldots + a_0).$$

The only thing that needs to be checked is whether im H lies in U or not. However, the triangle inequality and our assumption on c ensure

$$|a_{d-1}z^{d-1} + \ldots + a_0| < |z^d|$$

so $H(z,t)$ is never zero are we are done. □

14 Hairy coconut theorem

Aim lecture We compute the degree of the antipodal map to study vector fields on spheres.

Degree of the antipode

Below we consider the following triangulation of S^n. First let $\sigma = a_0 \ldots a_n$ be an n-simplex and $K = K_\sigma^{(n-1)}$ so radial projection from the barycentre $\hat{\sigma}$ of σ gives a triangulation $\bar{\theta} : |K| \xrightarrow{\sim} S^{n-1}$.

Consider the suspension $S(K) = K * w_+ \cup K * w_-$. We construct a triangulation $\theta : |S(K)| \longrightarrow S^n \subset \mathbb{R}^{n+1}_{x_0,\ldots,x_n}$ which extends the triangulation $\bar{\theta}$. Let

$$E_\pm := \{\vec{x} \in S^n | \pm x_n \geq 0\}$$

be the upper and lower hemispheres of S^n so the equator $E := E_+ \cap E_- \simeq S^{n-1}$. Place the simplex $\langle a_0 \ldots a_n w_+ \rangle$ in $\mathbb{R}^{n+1}$ with the base $|K|$ inscribed in the equator E, and w_+ at the north pole $(0, \ldots, 0, 1)$. DRAW PICTURE.

Radial projection from $\hat{\sigma}$ gives a homeomorphism $\theta_+ : |K * w_+| \longrightarrow E_+$ whch we may assume restricts to $\bar{\theta}$ on K. Similary, there is a homeomorphism $\theta_- : |K * W_-| \longrightarrow E_-$. They glue together to give the desired triangulation $\theta : |S(K)| \longrightarrow S^n$.

Lemma 14.1 *The radial projection map $r : \mathbb{R}^{n+1} - 0 \longrightarrow S^n : v \mapsto \frac{v}{|v|}$ commutes with reflection $\tau : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1} : (x_0, \dots, x_n) \mapsto (x_0 \dots, x_{n-1}, -x_n)$ about the $x_n = 0$ hyperpane.*

Proof. $r\tau(\vec{x}) = (x_0 \dots, x_{n-1}, -x_n)/|(x_0 \dots, x_{n-1}, \pm x_n)| = \tau r(\vec{x})$. □

Below we consider the group hom $[-w_\pm] : C_p(K) \longrightarrow C_p(K * w_\pm)$ defined on the free basis by $[a_{i_0} \dots a_{i_p}] \mapsto [a_{i_0} \dots a_{i_p} w_\pm]$.

Proposition 14.2 *Let $\tau|_{S^n} : S^n \longrightarrow S^n$ be reflection about a co-ordinate hyperplane. Then* $\deg \tau|_{S^n} = -1$.

Remark Since $\tau^2 = \text{id}$ and deg is multiplicative, we know $(\deg \tau|_{S^n})^2 = 1$.

Proof. We may assume that the hyperplane is $x_n = 0$. We first consider the vertex map $\phi : S(K)^{(0)} \longrightarrow S(K)^{(0)}$ which switches $w_\pm$ and leaves all other vertices fixed. Clearly this is a simplicial map and $|\phi| = \tau|_{|S(K)|} : |S(K)| \longrightarrow |S(K)|$. Moreover, from the construction of our triangulation $\theta = r|_{|S(K)|} : |S(K)| \longrightarrow S^n$ and lemma 14.1, we see that

$$\tau|_{S^n} = \theta|\phi|\theta^{-1} : S^n \xrightarrow{\theta^{-1}} |S(K)| \xrightarrow{|\phi|} |S(K)| \xrightarrow{\theta} S^n.$$

It thus suffices to show that $H_n(\phi)$ is multn by -1 (ex make sure you know why). To this end, let $z \in Z_n(S(K))$ be an n-cycle which must have the form

$$z = [z_+ w_+] + [z_- w_-]$$

for some $n-1$-chains $z_+, z_- \in C_{n-1}(K)$. Being cycle means

$$0 = \partial z = [\partial z_+ w_+] + (-1)^{n+1} z_+ + [\partial z_- w_-] + (-1)^{n+1} z_-.$$

We must thus have $\partial z_\pm = 0, z_+ + z_- = 0$. Hence $z = [z_+ w_+] - [z_+ w_-]$ and

$$\phi_* z = [z_+ w_-] - [z_+ w_+] = -z$$

and $H_n(\phi)$ must be multn by -1 as desired. □

Definition 14.3 *The* antipode *map is $a : S^n \longrightarrow S^n : v \mapsto -v$.*

Corollary 14.4 $\deg a = (-1)^{n+1}$.

Proof. Let $\tau_0, \dots, \tau_n : S^n \longrightarrow S^n$ be reflection about the co-ordinate hyperplanes $x_0 = 0, \dots, x_n = 0$. Then $a = \tau_0 \circ \dots \circ \tau_n$. Hence multiplicativity of the degree map and proposition 14.2 gives $\deg a = (-1)^{n+1}$. □

Fixed point theorems for spheres

Theorem 14.5 *Let $f : S^n \longrightarrow S^n$ be continuous. If $\deg f \neq (-1)^{n+1}$, then f has a fixed point.*

Proof. Suppose that f has no fixed point. By corollary 14.4, it suffices to show f is homotopic to the antipode a. In fact, the homotopy is $H : S^n \times I \longrightarrow S^n$ defined by

$$H(v,t) = \frac{(1-t)f(v) - tv}{|(1-t)f(v) - tv|}.$$

We need only check the denominator is never 0. Suppose instead that $(1-t)f(v) = tx$. Since $f(v), v$ both have length 1, we must have $t = \frac{1}{2}$. Then $f(v) = v$ giving the fixed point v, a contradiction. □

Vector fields on spheres

A continuous vector field v on the sphere S^n is a continuous map of the form $x \in S^n \mapsto v_x \in T_x S^n = x^\perp$ where $T_x S^n$ is the tangent space to S^n at x, which we identify with the subspace $x^\perp \subset \mathbb{R}^{n+1}$. We say v is *nowhere vanishing* if $v_x \neq 0$ for every x.

DRAW PICTURE

Theorem 14.6 *If S^n has a nowhere vanishing continuous vector field, then n is odd.*

Remark Explain name hairy coconut thm.

Proof. Suppose v is a nowhere vanishing continuous vector field. We can then define the continuous map $f : S^n \longrightarrow S^n$ by $x \mapsto \frac{v_x}{|v_x|}$. Since $x \perp v_x$ and $f(x)$, we know that f has no fixed point. Hence $\deg f = (-1)^{n+1}$ by theorem 14.5. Similarly, af has no fixed point so

$$(\deg a)(\deg f) = (-1)^{n+1} \quad \Longrightarrow \quad \deg f = 1.$$

This gives the equality $(-1)^{n+1} = 1$ so n must be odd. □

Addendum When n is odd, it is easy to construct continuous nowhere vanishing vector fields.

15 Singular homology

Aim lecture We introduce the homology functor on topological spaces. We will begin also our proof of the Main Theorem, and so in particular, we will not assume it until we've finally proved it.

Let X = topological space

Group of singular p-chains

Consider the infinite dimensional Euclidean space $\mathbb{R}^\infty$ and standard "basis" vectors

$$\mathbb{R}^\infty \ni \varepsilon_0 = (1,0,0,\ldots), \varepsilon_1 = (0,1,0,0,\ldots), \ldots$$

. The *standard (geometric) p-simplex* is

$$\Delta_p := \langle\, \varepsilon_0 \ldots \varepsilon_p \,\rangle\,.$$

A *singular p-simplex of* X is a continuous map $T : \Delta_p \longrightarrow X$.

E.g.

i. If $x \in X$, then the constant map $T_x : \Delta_0 \longrightarrow \{x\} \hookrightarrow X$ is a singular 0-simplex.

ii. The identity map $T_{\Delta_p} = \mathrm{id}_{\Delta_p} : \Delta_p \longrightarrow \Delta_p$ is a singular p-simplex.

iii. The affine linear map $l_i := l(\Delta_{p-1}, \langle\, \varepsilon_0 \dots \widehat{\varepsilon_i} \dots \varepsilon_p \,\rangle) : \Delta_{p-1} \longrightarrow \Delta_p$ is a singular $p-1$-simplex of Δ_p representing the i-th face.

iv. The affine linear map $l_{ij} := l(\Delta_{p-2}, \langle\, \varepsilon_0 \dots \widehat{\varepsilon_i} \dots \widehat{\varepsilon_j} \dots \varepsilon_p \,\rangle) : \Delta_{p-2} \longrightarrow \Delta_p$ is a singular $p-2$-simplex of Δ_p

Let $S_p(X)$ be the free abelian group generated by the singular p-simplices. The elements are called *singular p-chains*. Given a continuous map $f : X \longrightarrow Y$, consider the group homomophism $S_p(f) : S_p(X) \longrightarrow S_p(Y)$ defined on generators (i.e. p-simplices) by

$$S_p(f) : T \mapsto f \circ T =: f_\#(T).$$

Lemma 15.1 $S_p : \underline{Top} \longrightarrow \underline{Ab}$ *is a covariant functor.*

Proof. $S_p(\mathrm{id}\,) = \mathrm{id}$ since $S_p(\mathrm{id}\,)$ maps $T \mapsto \mathrm{id} \circ T = T$ so is the identity on generators. Also given continuous $X \xrightarrow{f} Y \xrightarrow{g} Z$ note $S_p(gf) = S_p(g)S_p(f)$ since on a p-simplex T, they both map $T \mapsto (gf)T = g(fT)$. □

Singular chain complex

We wish to assemble the $S_p(X)$ for $p \in \mathbb{Z}$ so that we obtain a chain complex and $S_\bullet$ becomes a functor $\underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Ch}}$. For a p-simplex T, its *i-th face* is the $(p-1)$-simplex

$$Tl_i : \Delta_{p-1} \xrightarrow{l_i} \Delta_p \xrightarrow{T} X.$$

DRAW PICTURE

The boundary operator is defined by

$$\partial_p : S_p(X) \longrightarrow S_{p-1}(X) : T \mapsto \sum_{i=0}^{p} (-1)^i T l_i.$$

Proposition 15.2 $\partial^2 = 0$ *so we have a complex*

$$S_\bullet(X) : \ldots \longrightarrow S_2(X) \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \longrightarrow 0$$

called the singular chain complex of X.

Proof. Note that affine linear maps are determined by their values on vertices, and that the composite of two is again affine linear. Hence $l_i l_j = l_{i,j+1}$ if $i \le j$ and $l_i l_j = l_{ji}$ if $j < i$. We compute ∂^2 on a p-simplex T.

$$\partial^2 T = \partial(\sum_i (-1)^i T l_i) = \sum_{i,j} (-1)^{i+j} T l_i l_j = \sum_{i<j'} (-1)^{i+j'-1} T l_{ij'} + \sum_{i>j} (-1)^{i+j} T l_{ji} = 0$$

where we split the sum and performed a change of variable $j' = j + 1$. □

Singular homology functor

Proposition 15.3 *Given a continuous map $f : X \longrightarrow Y$, the collection of maps $S_\bullet(f) = \{S_p(f)\} : S_\bullet(X) \longrightarrow S_\bullet(Y)$ is a chain map. This makes $S_\bullet : \underline{Top} \longrightarrow \underline{Ch}$ a covariant functor.*

Notn We also write f_* for $S_p(F)$.

Proof. The second assertion follows easily from the first by lemma 15.1. It thus suffices to check $\partial S_p(f) = S_{p-1}(f)\partial$. For a singular p-simplex T we have

$$\partial S_p(f)T = \partial(fT) = \sum_i (-1)^i (fT)l_i = S_{p-1}(f)(\sum_i (-1)^i T l_i) = S_{p-1}(f)\partial T.$$

□

Corollary 15.4 *We have a composite functor $H_p : \underline{Top} \xrightarrow{S_\bullet} \underline{Ch} \xrightarrow{H_p} \underline{Ab}$. It is called the* singular p-th homology functor *and $H_p(X)$ is the* p-th homology group of X.

We will show later that $H_p(f)$ only depends on the homotopy equivalence class of f so we will actually have a functor $H_p : \underline{\text{HTop}} \longrightarrow \underline{\text{Ab}}$.

Unlike simplicial homology, one usually cannot compute them from definitions because when X is infinite, all the $S_p(X)$ are infinitely generated.

Variants

Just as for simplicial homology, we can define reduced homology and homology with co-efficients in the setting of singular homology. Analogues of all the above results hold and we will not spell this out in future.

For reduced singular homology, we replace $S_\bullet(X)$ with the complex $\tilde{S}_\bullet(X)$, which is the same as $S_\bullet(X)$ except in degree -1 we have $\tilde{S}_{-1}(X) = \mathbb{Z}$ and the boundary map

$$\partial_0 : \tilde{S}_0(X) \longrightarrow \tilde{S}_{-1}(X) : T \mapsto 1$$

for any 0-simplex T. We then get the *p-th reduced homology of* X as $\tilde{H}_p(X) := H_p(\tilde{S}_\bullet(X))$.

Given a field $\mathbb{F}$, we define homology with co-efficients in $\mathbb{F}$, by replacing all the $S_p(X)$ with

$$S_p(X, \mathbb{F}) = \bigoplus_{p-\text{simplex } T} \mathbb{F}T$$

i.e. the vector space over $\mathbb{F}$ with basis the p-simplices of X. This gives functors $H_p(-, \mathbb{F}) : \underline{\text{Top}} \longrightarrow \underline{\text{Vect}_{\mathbb{F}}}$.

16 Acyclicity of simplices

Aim lecture We recover singular versions of basic results in simplicial homology e.g. we show that star convex top spaces have trivial reduced homology.

$$X = \text{topological space}$$

Bracket operation

For a simplicial cone $K * w$, we used the map $C_p(K) \longrightarrow C_{p+1}(K * w) : \sigma \mapsto \sigma\, w$. We need the topological analogue called the *bracket operation.*

We consider the $p+1$-st face $l_{p+1} : \Delta_p \longrightarrow \Delta_{p+1} : \varepsilon_i \mapsto \varepsilon_i$ which we view as the natural embedding. We extend this to a surjective continuous map $\phi : \Delta_p \times I \longrightarrow \Delta_{p+1} : (x, t) \mapsto (1-t)l_{p+1}x + t\,\varepsilon_{p+1}$. This identifies Δ_{p+1} as quotient space of $\Delta_p \times I$, more precisely, ϕ induces a homeomorphism $(\Delta_p \times I)/\sim \simeq \Delta_{p+1} : [(v,t)] \mapsto \phi(v,t)$ where the equivalence relation is generated by $(v,1) \sim (v',1)$ for all $v, v' \in \Delta_p$.

Suppose X is star convex relative to w. Given a p-simplex $T : \Delta_p \longrightarrow X$ we can construct a $p+1$-simplex $[T,w]$ as follows. Consider the continuous map $T_I : \Delta_p \times I \longrightarrow X : (v,t) \mapsto (1-t)T(v) + tw$. Because $T_I(v,1) = T_I(v',1)$, it induces a continuous map $[T,w] : \Delta_{p+1} \longrightarrow X$.

DRAW commutative diagram in class instead + picture.

Extending linearly, gives a group hom $[-,w] : \tilde{S}_p(X) \longrightarrow \tilde{S}_{p+1}(X)$. By default we set $\tilde{S}_{-1}(X) \longrightarrow \tilde{S}_0(X) : n \mapsto nT_w$.

Acyclicity

Lemma 16.1 *Let X be star convex rel to w.*

i. For a p-simplex T, $[T,w]l_i = [Tl_i, w]$ for $i \leq p$ (beware the l_i on the LHS is different from the l_i on the RHS).

ii. For $c \in \tilde{S}_p(X)$, we have

$$\partial[c,w] = [\partial c, w] + (-1)^{p+1}c.$$

Proof. (i) is an ex but is best understood from a picture.

(ii) By linearity of LHS & RHS, it suffices to note

$$\partial[T,w] = \sum_{i=0}^{p+1}(-1)^i[T,w]l_i = \sum_{i=0}^{p}(-1)^p[Tl_i,w] + (-1)^{p+1}[T,w]l_{p+1} = [\partial T, w] + (-1)^{p+1}T.$$

□

Definition 16.2 *We say X is* acyclic *if $\tilde{H}_p(X) = 0$ for all p.*

Theorem 16.3 *Let X be star convex rel to w. Then X is acyclic. In particular, any geometric simplex is acyclic.*

Proof. We show that any $c \in \tilde{Z}_p(X)$ lies in $\tilde{B}_p(X)$. Indeed

$$\partial[c,w] = [\partial c, w] + (-1)^{p+1}c = (-1)^{p+1}c$$

so $\partial(-1)^{p+1}[c,w] = c$. □

Compact support axiom

Proposition 16.4 *i. Let $\alpha \in H_p(X)$. Then there is a compact subspace $X_0 \subseteq X$, and $\alpha_0 \in H_p(X_0)$ such that $\iota_*(\alpha_0) = \alpha$ where $\iota : X_0 \hookrightarrow X$ is the inclusion.*

ii. Let $Y \subseteq X$ be a compact subspace and $\beta \in H_p(Y)$ be such that $\iota_*(\beta) = 0$ where $\iota : Y \hookrightarrow X$ is inclusion. Then there is a compact subspace $X_1 \subseteq X$ containing Y such that $\iota'_*\beta = 0$ where $\iota' : Y \hookrightarrow X_1$ is inclusion.

Proof. Both are similar, so we just prove (ii) (the harder one and the one we need). Let $b \in Z_p(Y)$ be a singular p-cycle representing β. Since $\iota_*\beta = 0$, we know there is a singular $p+1$-chain $d \in S_{p+1}(X)$ such that $b = \partial d$. Write $d = \sum_i n_i T_i$ for some singular p-simplices T_i and $n_i \in \mathbb{Z}$. Note that $\mathrm{im}\,(T_i : \Delta_{p+1} \longrightarrow X)$ is compact so $X_1 := Y \cup \bigcup_i \mathrm{im}\ T_i$ is too. This is the desired subspace as clearly $\iota'_*\beta = 0$ too. □

$\mathbf{H_0(X)}$

Just as for simplicial homology, H_0 measures the path connected components.

Proposition 16.5 *Let I be the set of path components of X and $\{x_i\}_{i \in I}$ be a set of representative points, i.e. x_i is a point in the path component i. Then*

$$H_0(X) \simeq \oplus_{i \in I}\, \mathbb{Z} T_{x_i}.$$

Proof. Let $A := \oplus_i \mathbb{Z} T_{x_i} \subseteq S_0(X) = Z_0(X)$. Note $A + B_0(X) = Z_0(X)$ since DRAW PICTURE.

Also $A \cap B_0(X) = 0$ since if $\sum_i n_i T_{x_i} \in B_0(X)$ then DRAW PICTURE

Then $H_0(X) \simeq A/(A \cap B_0(X)) = A$. □

17 Relative homology

Aim lecture We introduce a version of singular homology for topological pairs.

$$X = \text{topological space}, \quad (X, A) = \text{topological pair}$$

Quotient complexes

Let $C_\bullet$ be a complex and $C'_\bullet$ be a subcomplex. We define the *quotient complex* $C_\bullet/C'_\bullet$ to be the complex $C''_\bullet$ with $C''_p = C_p/C'_p$ and boundary maps ∂'' induced by the boundary map ∂ of $C_\bullet$, i.e.

$$\partial''_p : C''_p = C_p/C'_p \longrightarrow C''_{p-1} = C_{p-1}/C'_{p-1} : c + C'_p \mapsto \partial c + C'_{p-1}.$$

Note it is indeed a complex as $\partial^2 = 0 \implies \partial''^2 = 0$.

Suppose now we have a chain map $f_\bullet : C_\bullet \longrightarrow D_\bullet$, and that $D'_\bullet$ is a subcomplex of $D_\bullet$ such that $f_p(C'_p) \subseteq D'_p$. Then there is an induced chain map $\bar{f}_\bullet : C_\bullet/C'_\bullet \longrightarrow D_\bullet/D'_\bullet$ defined by

$$\bar{f}_p : C_p/C'_p \longrightarrow D_p/D'_p : c + C'_p \mapsto f(c) + D'_p.$$

It is readily seen to be well-defined and is a chain map because $f_\bullet$ is.

Relative homology

Note $S_\bullet(A)$ is a subcomplex of $S_\bullet(X)$ so we may define $S_p(X,A)$ to be the quotient complex $S_\bullet(X)/S_\bullet(A)$ and the (relative) *homology of* (X,A) to be

$$H_p(X,A) := H_p(S_\bullet(X,A)).$$

E.g. $H_p(X,\varnothing) = H_p(X)$.

Suppose $f:(X,A)\longrightarrow(Y,B)$ is a continuous map of pairs. Then $f_\#(S_p(A))\subseteq S_p(B)$ so there is an induced chain map $S_\bullet(X,A)\longrightarrow S_\bullet(Y,B)$ which induces group homomorphisms $f_* = H_p(f) : H_p(X,A)\longrightarrow H_p(Y,B)$. We easily obtain

Proposition 17.1 *$H_p : \underline{TopPair} \longrightarrow \underline{Ab}$ is a covariant functor.*

We may similarly define the *category of simplicial pairs* $\underline{\text{SimpPair}}$ whose objects consist of pairs (L,K) where L is a subcomplex of the simplicial complex K and a simplicial maps $f:(L,K)\longrightarrow(L',K')$ are simplicial maps $f:K\longrightarrow K'$ which restrict to simplicial maps $f:L\longrightarrow L'$. We can define $C_\bullet(L,K) = C_\bullet(K)/C_\bullet(L)$ and p-th relative homology by $H_p(L,K) = H_p(C_\bullet(L,K))$ and get *relative (simplicial) homology functor* $H_p\colon \underline{\text{SimpPair}} \longrightarrow \underline{\text{Ab}}$.

Exact sequences

Let $A_\bullet$ be a complex. We say that $A_\bullet$ is *exact* at A_p if $H_p(A_\bullet)=0$. If it is exact at every A_p, then we say $A_\bullet$ is *exact* or *acyclic.*

E.g. Given a subgroup $B<A$ with A abelian, we get an exact sequence

$$0\longrightarrow B \xrightarrow{\iota} A \xrightarrow{\pi} A/B \longrightarrow 0$$

where ι is inclusion and $\pi : a\mapsto a+B$. It's easy to check this.

The converse is the following:

Proposition 17.2 *Consider a complex of the form*

$$0\longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

(which just means $gf=0$).

i. Exactness at A means f is injective.

ii. Exacteness at C means g is surjective.

iii. Exactness of the sequence thus means that g induces an isomorphism $C\simeq B/f(A)$ via the first isomorphism theorem. In this case we say the complex is a short exact sequence *(SES for short).*

Proof. i) Exact at A iff $\ker f=0$ iff f is inj.

ii) Exact at C iff im $g=C$.

iii) Exact at B iff $\ker g = f(A)$, so first isomorphism thm completes the proof. □

The following is the famous 5-lemma.

Lemma 17.3 *Consider the commutative diagram of abelian groups and group homomorphisms below.*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{d_1} & A_2 & \xrightarrow{d_2} & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \xrightarrow{e_1} & B_2 & \xrightarrow{e_2} & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

Suppose that the two rows are exact and that f_1, f_2, f_4, f_5 are isomorphisms. Then f_3 is an isomorphism too.

Proof. This is done by "diagram chasing" and best shown on a board (see lectures). We'll briefly sketch the proof here, but it reads rather terse.

We'll show f_3 is injective, surjectivity being similar. Let $a_3 \in \ker f_3$. Let its image in A_4, B_3, B_4 be a_4, b_3, b_4 so $b_3 = b_4 = 0$. Now f_4 is bijective so $a_4 = 0$. Also exactness at A_3 means that there's some $a_2 \in A_2$ with $d_2(a_2) = a_3$. Now setting $b_2 := f_2(a_2)$ then $e_2(b_2) = f_3 d_2(a_2) = b_3 = 0$ so exactness at B_2 means there's some b_1 with $e_1(b_1) = b_2$. Bijectivity of f_1 means that we can find $a_1 \in A_1$ with $f_1(a_1) = b_1$ and f_2 bijective ensures $d_1(a_1) = a_2$. Hence $a_3 = d_2 d_1(a_1) = 0$. □

Long exact sequence in homology

Proposition 17.4 *Let (X, A) be a top pair. Let $i : A \hookrightarrow X$ & $\pi : X = (X, \varnothing) \longrightarrow (X, A)$ be inclusion maps. Then there is a* long exact sequence associated to (X, A)

$$\begin{aligned} \ldots \longrightarrow & H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{\pi_*} H_p(X, A) \\ \longrightarrow & H_{p-1}(A) \xrightarrow{i_*} H_{p-1}(X) \xrightarrow{\pi_*} H_{p-1}(X, A) \longrightarrow \ldots \end{aligned}$$

If $f : (X, A) \longrightarrow (Y, B)$ is a continuous map of pairs, then the maps $H_p(f)$ assemble to give a chain map between the two LES = (long exact sequences).

Proof. Follows from the LES proved in 18.1. □

E.g. Let (X, A) be a topological pair with A acyclic. Then LES $\implies H_p(X, A) \simeq H_p(X)$ for $p > 1$. If A and X are connected, then $H_1(X, A) \simeq H_1(X)$.

18 Long exact sequence

Aim lecture We relate the homologies of a complex with that of a subcomplex and the corresponding quotient.

$$C_\bullet, D_\bullet, E_\bullet \text{ are chain complexes with boundary maps } \partial_C, \partial_D, \partial_E$$

Short exact sequence of complexes

We say

$$0 \longrightarrow C_\bullet \xrightarrow{f_\bullet} D_\bullet \xrightarrow{g_\bullet} E_\bullet \longrightarrow 0$$

is a *short exact sequence* of chain complexes if for every p we have

$$0 \longrightarrow C_p \xrightarrow{f_p} D_p \xrightarrow{g_p} E_p \longrightarrow 0$$

is a SES of groups.

E.g. If $C'_\bullet$ is a subcomplex of $C_\bullet$ then the canonical inclusion and quotient maps give a SES of chain complexes

$$0 \longrightarrow C'_\bullet \longrightarrow C_\bullet \longrightarrow C_\bullet/C'_\bullet \longrightarrow 0.$$

A *morphism* of SES of chain complexes is a commutative diagram in $\underline{\text{Ch}}$ of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_\bullet & \xrightarrow{f'_\bullet} & D'_\bullet & \xrightarrow{g'_\bullet} & E'_\bullet & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\gamma} & & \big\downarrow & & \big\downarrow{\scriptstyle\varepsilon} & & \\ 0 & \longrightarrow & C_\bullet & \xrightarrow{f_\bullet} & D_\bullet & \xrightarrow{g_\bullet} & E_\bullet & \longrightarrow & 0 \end{array} \tag{1}$$

where the rows are SES of chain complexes.

Note that for any zero chain map $0_\bullet : C_\bullet \longrightarrow E_\bullet$ we have $H_p(0_\bullet) = 0$ so given a SES $0 \longrightarrow C_\bullet \longrightarrow D_\bullet \longrightarrow E_\bullet \longrightarrow 0$, we obtain a complex

$$0 \longrightarrow H_p(C_\bullet) \longrightarrow H_p(D_\bullet) \longrightarrow H_p(E_\bullet) \longrightarrow 0.$$

One might suspect or hope that this sequence is also exact. The truth (below) is both more complicated and infinitely more interesting.

The long exact sequence

Theorem 18.1 *Let* $0 \longrightarrow C_\bullet \xrightarrow{f_\bullet} D_\bullet \xrightarrow{g_\bullet} E_\bullet \longrightarrow 0$ *be a SES of chain complexes.*

i. There is a group hom $\partial_p : H_p(E_\bullet) \longrightarrow H_{p-1}(C_\bullet)$ *which is natural wrt morphisms of SESs of chain complexes. i.e. given a morphism as in (1), there is a commutative diagram.*

$$\begin{array}{ccc} H_p(E'_\bullet) & \xrightarrow{\partial'_p} & H_{p-1}(C'_\bullet) \\ \big\downarrow & & \big\downarrow \\ H_p(E_\bullet) & \xrightarrow{\partial_p} & H_{p-1}(C_\bullet) \end{array}$$

∂_p *is called the* connecting homomorphism *and is induced by* ∂_D*.*

ii. There is a long exact homology sequence

$$\begin{aligned} \ldots \longrightarrow & H_p(C_\bullet) \xrightarrow{f_*} H_p(D_\bullet) \xrightarrow{g_*} H_p(E_\bullet) \\ \xrightarrow{\partial_p} & H_{p-1}(C_\bullet) \xrightarrow{f_*} H_{p-1}(D_\bullet) \xrightarrow{g_*} H_{p-1}(E_\bullet) \longrightarrow \ldots \end{aligned}$$

Note that part i) and functoriality of H_p *mean that the whole long exact sequence is natural wrt morphisms of SESs of chain complexes.*

Remark This proves the LES for a top pair (X, A), proposition 17.4 on applying the theorem to the SES

$$0 \longrightarrow S_\bullet(A) \longrightarrow S_\bullet(X) \longrightarrow S_\bullet(X, A) \longrightarrow 0.$$

Proof. We spend the rest of this lecture proving this. Our first goal is to define the

Connecting homomorphism

This is best defined on a board using the diagram

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_p & \xrightarrow{f_p} & D_p & \xrightarrow{g_p} & E_p & \longrightarrow & 0 \\ & & \partial_{C,p}\downarrow & & \partial_{D,p}\downarrow & & \partial_{E,p}\downarrow & & \\ 0 & \longrightarrow & C_{p-1} & \xrightarrow{f_{p-1}} & D_{p-1} & \xrightarrow{g_{p-1}} & E_{p-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array} \quad . \tag{2}$$

Our definition initially depends on many choices. Let $e_p \in E_p$ be a p-cycle representing the homology class $[e_p] \in H_p(E_\bullet)$. We wish to define $\partial_p[e_p] \in H_{p-1}(C_\bullet)$. Exactness of the row at E_p means we can find $d_p \in D_p$ such that $g_p(d_p) = e_p$. Let $d_{p-1} = \partial_D(d_p)$. Since the RH square commutes, $g(d_{p-1}) = \partial_E(e_p) = 0$. Exactness of the row at D_{p-1} means that there's some $c_{p-1} \in C_{p-1}$ with $f(c_{p-1}) = d_{p-1}$. Exactness of the row at C_{p-1} ensures that c_{p-1} is uniquely determined by d_{p-1}. We define

$$\partial_p[e_p] = [c_{p-1}].$$

We check independence of the choice of d_p (here we leave e_p fixed). We could have changed it by $f(c_p)$ for some $c_p \in C_p$, that is, replaced d_p with $d_p + f(c_p)$. Since the LH square commutes, this just changes c_{p-1} by $\partial_C(c_p)$ which is a $(p-1)$-boundary. Hence the homology class $[c_{p-1}]$ is independent of the choice of d_p.

We need also check independence of the choice of e_p. We could have changed it by a p-boundary $\partial_E(e_{p+1})$ where $e_{p+1} \in E_{p+1}$. Picking d_{p+1} st $g(d_{p+1}) = e_{p+1}$ means d_p changes by $\partial_D(d_{p+1})$. This changes d_{p-1} by $\partial^2(d_{p+1}) = 0$.

This shows that ∂_p is well-defined, and by construction, it is additive i.e. a group hom.

To see naturality, we need to consider the 3-dimensional lattice of morphisms obtained by placing a primed version of (2) on top of (2). All squares of the cube commute and chasing through the defns shows naturality.

Proof of exactness

At $H_p(D_\bullet)$ We have already noted that $g_* f_* = 0$ so we need only check if $[d_p] \in \ker H_p(g_\bullet) = g_*$, then $[d_p] \in \operatorname{im} H_p(f_\bullet) = f_*$. Now $[d_p] \in \ker H_p(g_\bullet) \implies$ we can find e_{p+1} st $\partial(e_{p+1}) = g(d_p)$. Pick $d_{p+1} \in D_{p+1}$ st $g(d_{p+1}) = e_{p+1}$. Now commutativity of RH square means that $g(d_p - \partial(d_{p+1})) = e_p - e_p = 0$ so there's $c_p \in C_p$ with $f(c_p) = d_p - \partial(d_{p+1})$. We conclude

$$f_*[c_p] = [d_p - \partial(d_{p+1})] = [d_p].$$

At $H_p(C_\bullet)$ By construcion of the connecting hom, $f_*\partial_p = 0$ so it suffices to show given $[c_{p-1}] \in \ker f_*$, we can find $[e_p] \in H_p(E_\bullet)$ with $\partial_p[e_p] = [c_{p-1}]$. Now $[c_{p-1}] \in \ker f_* \implies$ there's $d_p \in D_p$ with $\partial_D(d_p) = f(c_{p-1})$ and if $e_p := g(d_p)$ is a p-cycle, then we are done as then $\partial_p[e_p] = [c_{p-1}]$. But $\partial_E(e_p) = g(\partial_D d_p) = gf(c_{p-1}) = 0$ so we're done.

At $H_p(C_\bullet)$ Similar proof left as ex.

□

19 Homotopy invariance

Aim lecture We show that homotopic maps induce the same map on homology so the homology functor gives a functor HTop $\longrightarrow$ Ab.

$$f_0, f_1 : X \longrightarrow Y \text{continuous}, \quad F : f_0 \approx f_1 : X \times I \longrightarrow Y \text{ homotopy}$$

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We define $h_0, h_1 : X \longrightarrow X \times I$ by $h_j(x) = (x, j)$ and note $H := \mathrm{id}_{X\times I} : X \times I \longrightarrow X \times I$ is a homotopy from h_0 to h_1. In fact, it is the universal homotopy out of X in the sense that $F = FH$ is the homotopy from $f_0 = Fh_0$ to $f_1 = Fh_1$. When $X = \Delta_p$ we use the more specialised notn $H^\Delta, h_0^\Delta, h_1^\Delta$.

Statement of theorem

Theorem 19.1 *With above notn, $f_{0*} = f_{1*} : H_p(X) \longrightarrow H_p(Y)$. Hence the homology functor can be viewed as a functor H_p : HTop $\longrightarrow$ Ab.*

Remark: There is a similar thm (with similar proof) for reduced homology and homology of a top pair.

Proof. By 7.6, it suffices to produce a chain homotopy between $f_{0\#} = S_\bullet(f_0)$ and $f_{1\#} = S_\bullet(f_1)$. We do this in the special case of the universal homotopy H above.

Lemma 19.2 *There are maps $D_X = D_{X,p} : S_p(X) \longrightarrow S_{p+1}(X \times I)$ such that*

i. D_X is a chain homotopy between $h_{0\#}, h_{1\#}$ i.e. For singular p-simplex $T : \Delta_p \longrightarrow X$,

$$\partial D_X T = h_{1\#}T - h_{0\#}T - D_X \partial T. \tag{3}$$

ii. D_X is natural in X, i.e. given cont $g : X \longrightarrow W$, there is a commutative square

$$\begin{array}{ccc} S_p(X) & \xrightarrow{D_X} & S_{p+1}(X \times I) \\ {\scriptstyle g_\#}\downarrow & & \downarrow{\scriptstyle (g\times I)_\#} \\ S_p(W) & \xrightarrow{D_W} & S_{p+1}(W \times I) \end{array}$$

Proof thm assuming lemma: The required chain homotopy is $D = F_\# D_X$ since the lemma gives

$$\partial D = \partial F_\# D_X = F_\# \partial D_X = F_\#(h_{1\#} - h_{0\#} - D_X \partial) = (Fh_1)_\# - (Fh_0)_\# - F_\# D_X \partial = f_{1\#} - f_{0\#} - D\partial.$$

$\square$

Proof lemma

Proof. This is proved by induction on p, using the *method of acyclic models*. For $p = 0$ we define

$$D_X T_x : \Delta_1 \simeq I \longrightarrow X \times I : t \mapsto (x, t).$$

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Inductive step Assume $D_{X,0} \dots D_{X,p-1}$ defined and satisfies lemma when defined. Key is to define $D_X T$ first for $X = \Delta_p$ & $T = T_{\Delta_p} = T_\Delta = \mathrm{id}_\Delta$. Consider $s_p \in S_p(\Delta_p \times I)$ given by

$$s_p = h^\Delta_{1\#} T_\Delta - h^\Delta_{0\#} T_\Delta - D_{\Delta_p}(\partial T_\Delta)$$

which is well-defined by induction. Eqn (3) means $r_p = D_{\Delta_p} T_\Delta$ is a solution to $\partial r_p = s_p$. By induction

$$\partial D_{\Delta_p}(\partial T_\Delta) = h^\Delta_{1\#}(\partial T_\Delta) - h^\Delta_{0\#}(\partial T_\Delta) - D_{\Delta_p}(\partial^2 T_\Delta).$$

Now s_p is a cycle because

$$\partial s_p = \partial h^\Delta_{1\#} T_\Delta - \partial h^\Delta_{0\#} T_\Delta - \left(h^\Delta_{1\#}(\partial T_\Delta) - h^\Delta_{0\#}(\partial T_\Delta) \right) = 0.$$

Now $\Delta_p \times I$ is convex, hence star convex and thus acyclic by theorem 16.3. Thus $H_p(\Delta_p \times I) = 0$ and the p-cycle s_p must be a boundary say $s_p = \partial r_p$. We define

$$D_{\Delta_p}(T_\Delta) = r_p$$

so eqn (3) holds for $X = \Delta_p, T = T_\Delta$.

Now for general X and $T = T\,\mathrm{id}_{\Delta_p} = T_\#(T_\Delta)$, we let

$$D_X T = (T \times \mathrm{id}_I)_\# D_{\Delta_p}(T_\Delta).$$

Proving parts i) & ii) are easy. For i) note that $(T \times \mathrm{id}_I) h^\Delta_j = h_j T$ so by the above we have

$$\begin{aligned} \partial D_X T &= \partial (T \times \mathrm{id}_I)_\# D_{\Delta_p}(T_\Delta) = (T \times \mathrm{id}_I)_\# \partial D_{\Delta_p}(T_\Delta) \\ &= (T \times \mathrm{id}_I)_\# \left(h^\Delta_{1\#} - h^\Delta_{0\#} - D_{\Delta_p}\partial \right)(T_\Delta) = \left(h_{1\#} T_\# - h_{0\#} T_\# \right)(T_\Delta) - D_X T_\# \partial (T_\Delta) \\ &= h_{1\#} T - h_{0\#} T - D_X \partial T \end{aligned}$$

and the second last equality follows by naturality in dimension $p - 1$.

For ii), let $g : X \longrightarrow W$ be continuous and note

$$D_W g_\#(T) = D_W(gT) = (gT \times \mathrm{id}_I)_\# D_{\Delta_p}(T_\Delta) = (g \times \mathrm{id}_I)_\# (T \times \mathrm{id}_I)_\# D_{\Delta_p}(T_\Delta) = (g \times \mathrm{id}_I)_\# D_X(T).$$

□

20 Barycentric subdivision operator

Aim lecture We introduce the useful tool of barycentric subdivision, which allows us to replace chains with arbitrarily fine chains.

$$X = \text{ topological space}$$

Basic idea

Let $a, b, c \in \mathbb{R}^N$. Then $T = l(\Delta_2, \langle\, abc \,\rangle)$ has boundary

$$\partial T = l(\Delta_1, \langle\, bc \,\rangle) - l(\Delta_1, \langle\, ac \,\rangle) + l(\Delta_1, \langle\, ab \,\rangle).$$

If a, b, c are collinear this can be pictured as

We can repeat this process as we like.

We wish to generalise this to higher dimensions.

Subdivision operator

We wish to define a chain map $\mathrm{sd}_{X\,\bullet} : \tilde{S}_\bullet(X) \longrightarrow \tilde{S}_\bullet(X)$. We do so by induction on degree with $\mathrm{sd}_{X\,p} = \mathrm{id}$ for $p = -1, 0$.

Let $p > 0$. We first look at the case $X = \Delta_p, T = T_\Delta \in S_p(X)$ and let w_p be the barycentre of Δ_p, i.e. the point with barycentric co-ords $\frac{1}{p+1}(1, 1, \ldots, 1)$. We define $\mathrm{sd}_p(T_\Delta) = (-1)^p[\mathrm{sd}_{p-1}(\partial T_\Delta), w_p]$ where the bracket operation here was defined in lecture 16.

For general X, we define $\mathrm{sd}_X : S_p(X) = \tilde{S}_p(X) \longrightarrow S_p(X)$ on singular p-chains by

$$\mathrm{sd}_X(T) = T_\#(\mathrm{sd}_{\Delta_p}(T_\Delta)).$$

Proposition 20.1 *sd_X is natural in X, i.e. given continuous $f : X \longrightarrow Y$, the following diagram commutes*

$$\begin{array}{ccc}
\tilde{S}_p(X) & \xrightarrow{sd_X} & \tilde{S}_p(X) \\
{\scriptstyle f_\#}\big\downarrow & & {\scriptstyle f_\#}\big\downarrow \\
\tilde{S}_p(Y) & \xrightarrow{sd_Y} & \tilde{S}_p(Y)
\end{array}$$

Proof. Let T be a singular p-simplex. Then

$$f_\#\mathrm{sd}_X(T) = f_\# T_\# \mathrm{sd}_{\Delta_p}(T_\Delta) = (fT)_\#\mathrm{sd}_{\Delta_p}(T_\Delta) = \mathrm{sd}_Y(fT) = \mathrm{sd}_Y f_\# T.$$

□

Proposition 20.2 *sd_X is a chain map.*

Proof. We prove $\mathrm{sd}_{p-1}\partial_p = \partial_{p-1}\mathrm{sd}_p$ by induction on p. For $p \leq 0$ this is easily checked. Given T a p-simplex, we compute using the inductive hypothesis and naturality

$$\begin{aligned}
\partial\mathrm{sd}(T) = \partial(-1)^p T_\#[\mathrm{sd}\partial T_\Delta, w_p] &= (-1)^p T_\# \partial[\mathrm{sd}\partial T_\Delta, w_p] \\
&= (-1)^p T_\# \left([\partial\mathrm{sd}\partial T_\Delta, w_p] + (-1)^p \mathrm{sd}\partial T_\Delta\right) \\
&= (-1)^p T_\# \left([\mathrm{sd}\partial^2 T_\Delta, w_p] + (-1)^p \mathrm{sd}\partial T_\Delta\right) \\
&= \mathrm{sd}_X T_\# \partial T_\Delta = \mathrm{sd}_X \partial T
\end{aligned}$$

□

Diameter of subdivisions

Let $\langle \sigma \rangle = \langle a_0 \dots a_p \rangle$ be a geometric p-simplex and $T_\sigma = l(\Delta_p, \langle \sigma \rangle)$ be the corresponding singular p-simplex. Then $\mathrm{sd}^m T_\sigma$ will be a sum of p-simplices T and we call $\mathrm{im}\, T$ a geometric simplex of $\mathrm{sd}^m \sigma$ (note we haven't defined the latter, but we won't need to).

Proposition 20.3 *The diameters (wrt say the $|\cdot|_\infty$ norm) of the geometric simplices of $\mathrm{sd}^m \sigma$ tend to 0 as $m \longrightarrow \infty$.*

Proof. We omit the tedious but elementary proof of this fact which is readily seen from any picture.

The key fact is that the simplices of $\mathrm{sd}\, \sigma$ have diameter at most $\frac{p}{p+1}$ that of σ. □

Admissible covers and subdivision

Let X be a top space and $\mathcal{A}$ be a collection of subsets. We say $\mathcal{A}$ is an *admissible cover* if the interiors of $A \in \mathcal{A}$ form an open cover of X. We say a singular p-chain $c = \sum_i n_i T_i$ (with all $n_i \neq 0$) is $\mathcal{A}$-*small* if there are $A_i \in \mathcal{A}$ such that $\mathrm{im}\, T_i \subseteq A_i$.

Hopefully, you know the following from MATH3611.

Lemma 20.4 *Let X be a compact metric space and $\mathcal{U}$ be an open cover. There is a positive real number λ (called the* Lebesgue number for covers*) such that any subset $Z \subseteq X$ with diameter $< \lambda$ is contained in some $U \in \mathcal{U}$.*

Proof. If no such λ exists, then for each $n \in \mathbb{N}$, we can find a subset Z_n with diameter $< \frac{1}{n}$ which does not lie in any $U \in \mathcal{U}$. Pick $z_n \in Z_n$. By compactness, the sequence z_n has convergent subsequence, whose limit is say $z \in X$. Pick $U \in \mathcal{U}$ containing z so there is some ε-ball B_z centred at z which lies in U. We obtain a contradiction as for $n \gg 0$ we must have $Z_n \subseteq B_z$. □

Proposition 20.3 and this lemma give

Proposition 20.5 *Let $\mathcal{A}$ be an admissible cover of X. For any p-simplex T of X, $\mathrm{sd}^m T$ is $\mathcal{A}$-small for m large enough.*

Proof. We may assume $\mathcal{A}$ is an open cover by replacing each $A \in \mathcal{A}$ with its interior. Then $\{T^{-1}(U) | U \in \mathcal{A}\}$ is an open cover of Δ_p with say, Lebesgue number λ. Any m such that the diameters of the geometric simplices of $\mathrm{sd}^m \Delta_p$ are $< \lambda$ will do. □

21 Excision

Aim lecture We prove the excision theorem which allows cutting & pasting type arguments.

$$X = \text{topological space}, \mathcal{A} = \text{admissible cover}$$

Subdivision is homotopy equivalent to the identity

Theorem 21.1 *Fix m. For each p, there is a natural transformation $D : \tilde{S}_p \longrightarrow \tilde{S}_{p+1}$ which induces a chain homotopy between sd^m and id, i.e. for any p-simplex T of X, we have*

$$\partial D_X T + D_X \partial T = sd_X^m T - T. \tag{4}$$

Proof. (Sketch) We use the method of acyclic models as in Lemma 19.2. We use induction on p with $D_X = 0$ if $p \leq 0$.

Assume D_X defined satisfying (4) and naturality for d-simplices, $d < p$. We next define $D_X T$ for $X = \Delta_p, T = T_\Delta$ so (4) holds:

$$\partial D_{\Delta_p} T_\Delta + D_{\Delta_p} \partial T_\Delta = \mathrm{sd}^m T_\Delta - T_\Delta. \tag{5}$$

Now Δ_p is acyclic, so this is possible so long as

$$z_p := -D_{\Delta_p} \partial T_\Delta + \mathrm{sd}^m T_\Delta - T_\Delta$$

is a cycle. This follows by a simple computation using induction.

We can now define for general X, T, $D_X T = T_\# D_{\Delta_p} T_\Delta$. Naturality follows readily and applying $T_\#$ to (5) using proposition 20.1 gives (4). □

$\mathcal{A}$-small chain complex

We let $\tilde{S}_p^{\mathcal{A}}(X)$ be the subgroup of $\tilde{\mathrm{S}}_p(X)$ generated by $\mathcal{A}$-small simplices. Note that these define a subcomplex $\tilde{\mathrm{S}}_\bullet^{\mathcal{A}}(X)$. For a topological pair (X, B) we define $S_\bullet^{\mathcal{A}}(X, B) = \tilde{\mathrm{S}}_\bullet^{\mathcal{A}}(X) / \tilde{\mathrm{S}}_\bullet^{\mathcal{A}}(B)$.

Lemma 21.2 *The quotient chain complex $\bar{S}_\bullet(X) := \tilde{\mathrm{S}}_\bullet(X) / \tilde{\mathrm{S}}_\bullet^{\mathcal{A}}(X)$ is acyclic.*

Proof. Let $c_p + \tilde{\mathrm{S}}_p^{\mathcal{A}}(X)$ be a p-cycle in $\bar{S}_p(X)$. This means that $\partial c_p \in \tilde{\mathrm{S}}_{p-1}^{\mathcal{A}}$. Pick m large enough so $\mathrm{sd}_X^m c_p \in \tilde{\mathrm{S}}_p^{\mathcal{A}}(X)$.

We use the chain homotopy D_X of theorem 21.1 between sd_X^m and id. Then

$$\partial D_X c_p = -D_X \partial c_p + \mathrm{sd}_X^m c_p - c_p.$$

Now naturality of D_X ensures that $D_X \partial c_p \in \tilde{\mathrm{S}}_p^{\mathcal{A}}(X)$ too so

$$\partial(D_X c_p + \tilde{\mathrm{S}}_p^{\mathcal{A}}(X)) = -c_p + \tilde{\mathrm{S}}_p^{\mathcal{A}}(X).$$

Hence $c_p + \tilde{\mathrm{S}}_p^{\mathcal{A}}(X)$ is also a boundary. □

Theorem 21.3 *The inclusion $\iota_\bullet : \tilde{\mathrm{S}}_\bullet^{\mathcal{A}}(X) \longrightarrow \tilde{\mathrm{S}}_\bullet(X)$ chain map induces an isomorphism on homology.*

Proof. We use the long exact homology sequence on

$$0 \longrightarrow \tilde{\mathrm{S}}_\bullet^{\mathcal{A}}(X) \longrightarrow \tilde{\mathrm{S}}_\bullet(X) \longrightarrow \tilde{\mathrm{S}}_\bullet(X) / \tilde{\mathrm{S}}_\bullet^{\mathcal{A}}(X) \longrightarrow 0$$

which, by acyclicity proven in the lemma, is

$$0 \longrightarrow H_p(\tilde{\mathrm{S}}_\bullet^{\mathcal{A}}(X)) \xrightarrow{\iota_*} H_p(\tilde{\mathrm{S}}_\bullet(X)) \longrightarrow 0.$$

The theorem follows as then ι_* is both injective and surjective. □

Remark: There is a similar theorem for usual homology with similar proof.

Corollary 21.4 *For a topological pair (X, B), the inclusion $\iota_\bullet : S_\bullet^{\mathcal{A}}(X, B) \longrightarrow S_\bullet(X, B)$ induces an isomorphism on homology.*

Proof. Consider the map of SES of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathrm{S}}^{\mathcal{A}}_\bullet(B) & \longrightarrow & \tilde{\mathrm{S}}^{\mathcal{A}}_\bullet(X) & \longrightarrow & \tilde{\mathrm{S}}^{\mathcal{A}}_\bullet(X,B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{\mathrm{S}}_\bullet(B) & \longrightarrow & \tilde{\mathrm{S}}_\bullet(X) & \longrightarrow & \tilde{\mathrm{S}}_\bullet(X,B) & \longrightarrow & 0 \end{array}$$

There is a corresponding map on LES in homology which by the previous theorem are isomorphisms on the homologies of B and X. The 5-lemma gives the corollary. □

Excision theorem

Theorem 21.5 *Let (X,A) be a top pair, and $U \subset A$ be such that $\bar{U}$ is in the interior $IntA$ of A. Then the inclusion $(X-U, A-U) \longrightarrow (X,A)$ induces an isomorphism in homology.*

Proof. We consider the admissible cover $\mathcal{A} = \{X-U, A\}$. By corollary 21.4, it suffices to prove that the natural maps

$$\frac{S_p(X-U)}{S_p(A-U)} \longrightarrow \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)}$$

are isomorphisms. This follows from the subgroup isomorphism theorem since $S_p(X-U) + S_p^{\mathcal{A}}(A) = S_p^{\mathcal{A}}(X)$ and $S_p(X-U) \cap S_p^{\mathcal{A}}(A) = S_p(A-U)$. □

22 Mayer-Vietoris sequence. Ordered homology

Aim lecture We prove the Mayer-Vietoris sequence which allows us to compute the singular homology of spheres and other topological spaces (without resort to our main theorem). We introduce ordered homology of simplicial complexes in preparation for the proof of the main theorem 12.2.

$$X = \text{topological space}, K = \text{ simplicial complex}$$

Mayer-Vietoris

Theorem 22.1 *Let $\mathcal{A} = \{X_1, X_2\}$ be an admissible cover for X and $A = X_1 \cap X_2$. Then there is an exact (Mayer-Vietoris) sequence*

$$\ldots \longrightarrow \tilde{\mathrm{H}}_p(A) \xrightarrow{f_*} \tilde{\mathrm{H}}_p(X_1) \oplus \tilde{\mathrm{H}}_p(X_2) \xrightarrow{g_*} \tilde{\mathrm{H}}_p(X) \longrightarrow \tilde{\mathrm{H}}_{p-1}(A) \longrightarrow \ldots$$

Here $f_*(a) = (i_{1*}a, -i_{2*}a)$ where $i_l : A \longrightarrow X_l$ is inclusion and $g_*(x_1,x_2) = j_{1*}(x_1) + j_{2*}(x_2)$ where $j_l : X_l \longrightarrow X$ are inclusions.

Remark: There is a similar theorem for homology.

Proof. Note that $\tilde{\mathrm{S}}^{\mathcal{A}}_\bullet(X) = \tilde{\mathrm{S}}_\bullet(X_1) + \tilde{\mathrm{S}}_\bullet(X_2)$, so the theorem follows from the LES in homology applied to the SES

$$0 \longrightarrow \tilde{\mathrm{S}}_\bullet(A) \xrightarrow{f_\#} \tilde{\mathrm{S}}_\bullet(X_1) \oplus \tilde{\mathrm{S}}_\bullet(X_2) \xrightarrow{g_\#} \tilde{\mathrm{S}}_\bullet(X_1) + \tilde{\mathrm{S}}_\bullet(X_2) \longrightarrow 0$$

and theorem 21.3. (Ex can you work out the chain maps $f_\#, g_\#$ above which induce f_*, g_*.) You also need to know that the homology of the direct sum of complexes is the direct sum of the homologies. □

Suspensions

Definition 22.2 *The* suspension of X *is the topological space* $SX = (X \times I)/\sim$ *where* $\sim$ *is the weakest equivalence relation with* $(x,0) \sim (x',0)$ *and* $(x,1) \sim (x',1)$ *for* $x, x' \in X$.

Example 22.3 $S(S^{n-1}) \simeq S^n$.

Proof. Best seen by picture. The homeomorphism is induced by the continuous map $\phi : S^{n-1} \times [-1,1] \longrightarrow S^n$ defined as follows:

$$\phi(\vec{x}, t) = ((1-t^2)\vec{x}, t) \in \mathbb{R}^{n+1}.$$

□

Theorem 22.4 $\tilde{\mathrm{H}}_p(X) \simeq \tilde{\mathrm{H}}_{p+1}(SX)$.

Proof. Note that $\{X \times (0,1], X \times [0,1)\}$ is an open cover of $X \times I$ which induces an open cover $\{X_1, X_2\}$ of SX. The Mayer-Vietoris sequence gives the exact sequence

$$\tilde{\mathrm{H}}_{p+1}(X_1) \oplus \tilde{\mathrm{H}}_{p+1}(X_2) \longrightarrow \tilde{\mathrm{H}}_{p+1}(SX) \longrightarrow H_p(X \times (0,1)) \longrightarrow \tilde{\mathrm{H}}_{p+1}(X_1) \oplus \tilde{\mathrm{H}}_{p+1}(X_2). \quad (6)$$

Note that $X \times (0,1)$ is homotopy equivalent to X since $(0,1)$ is homotopy equivalent to a point.

Also $X \times 1$ is a weak deformation retract of $X \times (0,1]$ so one easily sees that X_1 is homotopy equivalent to a point and hence acyclic. Similarly X_2 is acyclic so the outer groups in (6) are 0 and the theorem is proved. □

Homology of spheres

Proposition 22.5 *For* $n \geq 1$ *we have*

$$H_p(S^n) = \begin{cases} 0 & \textit{if } p \neq 0 \textit{ or } n \\ \mathbb{Z} & \textit{if } p = 0 \textit{ or } n \end{cases}.$$

Proof. Since S^n is connected, it suffices to prove $\tilde{\mathrm{H}}_p(S^n) \simeq \mathbb{Z}$ when $p = n$ and is 0 otherwise. By theorem 22.4, it suffices to check $\tilde{\mathrm{H}}_0(S^0 = 2 \text{ points}) = \mathbb{Z}$. This is easily computed from the reduced singular chain complex

$$\mathbb{Z} \oplus \mathbb{Z} = \tilde{\mathrm{S}}_1(S^0) \xrightarrow{0} \tilde{\mathrm{S}}_0(S^0) = \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \tilde{\mathrm{S}}_{-1}(S^0) = \mathbb{Z} \longrightarrow 0.$$

□

Remark: Note that this allows one to prove (without recourse to our main theorem 12.2), the Brouwer fixed point theorem 13.4, and the fact (theorem 12.5) that $\mathbb{R}^m \not\simeq \mathbb{R}^n$ for $m \neq n$.

Ordered homology

Let K = simplicial complex. An *ordered p-simplex* is a $p+1$-tuple $(a_0, \ldots, a_p)$ where $a_0, \ldots, a_p$ are the not necessarily distinct vertices of a simplex of K.

Let $C_p^<(K)$ be the free abelian group generated by the ordered p-simplices. There's a boundary map

$$\partial : C_p^<(K) \longrightarrow C_{p-1}^<(K) : (a_0, \ldots, a_p) \mapsto \sum_{i=0}^{p} (-1)^p (a_0, \ldots, \widehat{a_i}, \ldots, a_p)$$

which makes $C_\bullet^<(K)$ a complex. We define the *p-th ordered homology group* to be $H_p^<(K) := H_p(C_\bullet^<(K))$. Reduced ordered homology $\tilde{\mathrm{H}}_p^<(K)$ is defined similarly.

Many of the standard proofs for usual oriented homology carry over to this setting such as

Proposition 22.6 $\tilde{\mathrm{H}}_p^<(K * w) = 0$.

Relative homology can also be defined in this setting as well as the usual setting: given a subcomplex L of K, we define

$$H_p^<(K, L) = H_p(C_\bullet^<(K)/C_\bullet^<(L))$$

and similarly for $H_p(K, L)$.

The rest of this lecture is not examinable. We now prove that the usual oriented and ordered homology are naturally isomorphic. Ideally, we'd like to construct natural transformations $\phi : C_\bullet \longrightarrow C_\bullet^<, \psi : C_\bullet^< \longrightarrow C_\bullet$.

$$\phi_{K,p} : C_p(K) \longrightarrow C_p^<(K) : a_0 \dots a_p \mapsto (a_0, \dots, a_p)$$

$$\psi_{K,p} : C_p^<(K) \longrightarrow C_p(K) : (a_0, \dots, a_p) \mapsto \begin{cases} [a_0, \dots, a_p] & \text{if } a_0, \dots, a_p \text{ are distinct} \\ 0 & \text{otherwise} \end{cases}$$

One checks easily that ϕ_K, ψ_K are chain maps. Also, ψ_K is natural in K though ϕ_K unfortunately, is not.

Theorem 22.7 *i.* $\psi_K \phi_K = id_{C_\bullet(K)}$

ii. $\phi_K \psi_K$ *is homotopy equivalent to the identity chain map on* $C_\bullet^<(K)$.

iii. ϕ_K, ψ_K *induce inverse isomorphisms on homology so in particular,* $\phi_{K*} : H_p(K) \simeq H_p^<(K)$. *In other words,* ϕ_* *is a natural isomorphism of functors* $H_p \simeq H_p^< : \underline{Simp} \longrightarrow \underline{Ab}$.

Proof. We need only prove (ii) since (i) is immediate and (i) & (ii) $\implies$ (iii). The proof of (ii) follows from a variant of the method of acyclic models called the method of *acyclic carriers*. It works because cones are acyclic in ordered homology.

For completeness, we include a sketch of this proof. We wish to construct the chain homotopy $s_\bullet$ between $\psi_K \phi_K$ and id. Suppose by induction that $s_0, \dots, s_{p-1}$ have been constructed so that

$$\partial s + s\partial = \psi_K \phi_K - \mathrm{id} \qquad (7)$$

when defined and furthermore, that if $a_0, \dots, a_n$ are the vertices of a simplex σ, then $s_n(a_0, \dots, a_n) \in C_{n+1}^<(K_\sigma)$. Let $(a_0, \dots, a_p)$ be an ordered p-simplex and $\sigma \in K$ be the simplex spanned by the a_i. We wish to define $s(a_0, \dots, a_p)$ so equation (7) holds on $(a_0, \dots, a_p)$ and $s(a_0, \dots, a_p) \in C_{p+1}^<(K_\sigma)$. For this, note first that $\psi_K \phi_K$ preserves the chain subcomplex $C_\bullet^<(K_\sigma)$ which has trivial homology in degrees $p > 0$. Hence we can solve $\partial s(a_0, \dots, a_p) + s\partial(a_0, \dots, a_p) = \psi_K \phi_K(a_0, \dots, a_p) - (a_0, \dots, a_p)$ for $s(a_0, \dots, a_p)$ provided

$$c = -s\partial(a_0, \dots, a_p) + \psi_K \phi_K(a_0, \dots, a_p) - (a_0, \dots, a_p)$$

is a p-cycle. But by induction

$$\begin{aligned} \partial c &= -\partial s \partial(a_0, \dots, a_p) + \psi_K \phi_K \partial(a_0, \dots, a_p) - \partial(a_0, \dots, a_p) \\ &= s\partial^2(a_0, \dots, a_p) - \psi_K \phi_K \partial(a_0, \dots, a_p) + \partial(a_0, \dots, a_p) + \psi_K \phi_K \partial(a_0, \dots, a_p) - \partial(a_0, \dots, a_p) = 0 \end{aligned}$$

□

23 Compatibility of simplicial and singular homology

Aim lecture We finally prove our main theorem as stated in 12.2.

$$K = \text{simplicial complex}, \quad X = \text{topological space}$$

Ordered and singular homology

Now we wish to construct a natural transformation $\theta : C_\bullet^< \longrightarrow S_\bullet(|\cdot|)$ of functors $\underline{\text{Simp}} \longrightarrow \underline{\text{Ch}}$ such that on composing with $H_p : \underline{\text{Ch}} \longrightarrow \underline{\text{Ab}}$ gives a natural isomorphism between ordered homology and singular homology of the polytope. Together with theorem 22.7, this will complete the proof of theorem 12.2.

We define $\theta_K : C_\bullet^<(K) \longrightarrow S_\bullet(|K|)$ on ordered p-simplices to be

$$\theta_{K,p} : C_p^<(K) \longrightarrow S_p(K) : (a_0, \ldots, a_p) \mapsto l(\Delta_p, \langle a_0, \ldots, a_p \rangle).$$

Comparing the two similarly defined boundary maps in $C_\bullet^<$ and $s_\bullet$, we see θ_K is indeed a chain map and naturality is also easily checked from the formulas.

We may similarly define a chain map $\theta_K : \tilde{C}_\bullet^<(K) \longrightarrow \tilde{\text{S}}_\bullet(|K|)$ (let $\theta_{K,-1} = \text{id}_{\mathbb{Z}}$) and for a subcomplex L of K, θ_K induces a chain map $\theta_{K,L} : \tilde{C}_\bullet^<(K, L) \longrightarrow \tilde{\text{S}}_\bullet(|K|, |L|)$.

Lemma 23.1 *θ_{K*} is an isomorphism in reduced homology (for all p) iff it's an isomorphism in homology.*

Proof. Let $\mathbb{Z}_{-1}$ be the chain complex with a single $\mathbb{Z}$ in degree -1 and 0's elsewhere. We obtain a morphism of SES of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_\bullet^<(K) & \longrightarrow & \tilde{C}_\bullet^<(K) & \longrightarrow & \mathbb{Z}_{-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_\bullet(|K|) & \longrightarrow & \tilde{S}_\bullet(|K|) & \longrightarrow & \mathbb{Z}_{-1} & \longrightarrow & 0 \end{array}$$

There is a map on the LES in homology and the 5-lemma completes the proof. □

Theorem 23.2 *Let L be a subcompex of K. Then the chain maps θ_K (together with ϕ_K in theorem 22.7) induce the following isomorphisms on homology:*

i. $H_p(K) \simeq H_p(|K|)$.

ii. $\tilde{\text{H}}_p(K) \simeq \tilde{\text{H}}_p(|K|)$.

iii. $H_p(K, L) \simeq H_p(|K|, |L|)$.

Proof. The proof is by induction on the number n of simplices on K, the case where $n = 0$ or 1 being easily checked. We assume the theorem proved complexes with fewer than n simplices. It suffices to prove (1) for (2) follows from lemma 23.1, and (3) is trivial when $L = K$ and follows from induction, the LES and 5-lemma when $L \neq K$.

We now prove (1). Pick a p-simplex $\sigma \in K$ with p maximal. Let K_0 be the subcomplex of K consisting of all simplices of K except σ. Note maximality of p ensures K_0 is indeed still a complex.

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Note that (1) holds for $K = K_\sigma$ since in this case, the reduced homology in both the ordered simplicial case, and the singular case is zero. Hence by the 5-lemma and the LES for the simplicial pair (K, K_σ) and the topological pair $(|K|, |K_\sigma|)$, it suffices to show θ_{K,K_σ} induces an isomorphism in homology.

We consider the natural inclusion map of simplicial pairs $\iota : (K_0, K_\sigma^{(p-1)}) \longrightarrow (K, K_\sigma)$ and the corresponding inclusion of topological pairs $|\iota| : (|K_0|, |K_\sigma^{(p-1)}|) \longrightarrow (|K|, |K_\sigma|)$. Naturality of θ gives the following commutative diagram

$$\begin{array}{ccc} H_p(K_0, K_\sigma^{(p-1)}) & \xrightarrow{\iota_*} & H_p(K, K_\sigma) \\ \theta_{K_0*}\big\downarrow & & \theta_{K*}\big\downarrow \\ H_p(|K_0|, |K_\sigma^{(p-1)}|) & \xrightarrow{|\iota|_*} & H_p(|K|, |K_\sigma|) \end{array}$$

By induction, we know that θ_{K_0*} above is an isomorphism, so it suffices to show that ι_* and $|\iota|_*$ are too.

Now ι_* is an isomorphism by "excision in ordered homology" which is easy to see as follows. Note that the subgroup isomorphism theorem shows that

$$\frac{C_p^<(K_0)}{C_p^<(K_\sigma^{(p-1)})} \simeq \frac{C_p^<(K)}{C_p^<(K_\sigma)}.$$

Hence $\iota_\# : C_\bullet^<(K_0, K_\sigma^{(p-1)}) \longrightarrow C_\bullet^<(K, K_\sigma)$ is an isomorphism of chain complexes so induces an isomorphism on homology.

Although we can't use excision to conclude immediately that $|\iota|_*$ is an isomorphism, we can modify the situation to do so in the following argument I'll refer to as "excision + ". Pick a point x in the interior of $\langle \sigma \rangle$.

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We factorise $|\iota|$ into the composite of two inclusion maps

$$|\iota| : (|K_0|, |K_\sigma^{(p-1)}|) \xrightarrow{\iota'} (|K| - x, |K_\sigma| - x) \xrightarrow{\iota''} (|K|, |K_\sigma|).$$

Note that the excision theorem does apply to ι'' since the closure of x is contained in the interior of $|K_\sigma|$. Hence ι''_* is an isomorphism in homology. Also, by the LES in homology (see Q4 of Problem Set 4), it suffices to show that the inclusion maps $\iota' : |K_0| \longrightarrow |K| - x$ and its restriction $\iota' : |K_\sigma^{(p-1)}| \longrightarrow |K_\sigma| - x$ are homotopy equivalences and so induce isomorphisms on homology. The homotopy inverses are easiliest seen from a picture.

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This concludes the proof of the theorem. □

24 Jordan curve theorem

Aim lecture We prove the intuitively obvious fact that a simple closed plane curve has an inside and outside.

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Topological preliminaries

Recall that a map of topological spaces $f : X \longrightarrow Y$ is an *imbedding* if it is an homeomorphism with its image. If X is compact and Y Hausdorff, this amounts to f being continuous and injective. In particular, a *simple closed curve* in $\mathbb{R}^2$ is just an imbedding $S^1 \longrightarrow \mathbb{R}^2$.

Recall that if Y is locally path connected (e.g. Y = open subset of $\mathbb{R}^n, S^n$ or any manifold), then all path components are open and hence the path components and connected components agree.

We restate the classical Jordan curve theorem by adding a point at ∞ to $\mathbb{R}^2$ to get S^2.

Theorem 24.1 *Let $f : S^1 \longrightarrow S^2$ be an imbedding. Then $S^2 - f(S^1)$ has two connected components.*

We prove a more general version later. The component containing ∞ is the "outside" of $f(S^1)$ and the other is the "inside".

The statement of the theorem involves only H_0 but the proof naturally involves all the homology groups as we shall see.

Key lemma

Lemma 24.2 *Let $f : B^m \longrightarrow S^n$ be an imbedding. Then $S^n - f(B^m)$ is acyclic.*

Proof. We prove this by induction on m, the case $m = 0$ follows since $S^n -$ pt is homeomorphic to $\mathbb{R}^n$.

Note that the unit cube $I^m \simeq B^m$, so we may replace B^m above with I^m. Given any subset $Z \subseteq I$, we let $B_Z = f(I^{m-1} \times Z)$ so for example, $B_{\frac{1}{2}}$ denotes an embedded B^{m-1} and $B_I = \text{im } f$.

Let $X_0 = S^n - B_{[0,\frac{1}{2}]}, X_1 = S^n - B_{[\frac{1}{2},1]}$ so that

$$A := X_0 \cap X_1 = S^n - B_I, \quad X := X_0 \cup X_1 = S^n - B_{\frac{1}{2}}.$$

Note $\{X_0, X_1\}$ is an open and hence admissible cover for X so the Mayer-Vietoris sequence gives the exact sequence

$$\tilde{\mathrm{H}}_{p+1}(X) \longrightarrow \tilde{\mathrm{H}}_p(A) \xrightarrow{(i_{0*},i_{1*})} \tilde{\mathrm{H}}_p(X_0) \oplus \tilde{\mathrm{H}}_p(X_1) \longrightarrow \tilde{\mathrm{H}}_p(X)$$

By induction, the outer groups are 0 so (i_{0*}, i_{1*}) is an isomorphism where $i_j : A \hookrightarrow X_j$ are the inclusion maps. Suppose by way of contradiction, that $c \in \tilde{\mathrm{H}}_p(A)$ is non-zero. Then either $i_{0*}c \neq 0$ or $i_{1*}c \neq 0$. In the first case we pick $Z_1 = [0, \frac{1}{2}]$ and in the second, we pick $Z_1 = [\frac{1}{2}, 1]$ (we can pick either half interval if both homology classes are non-zero).

Now B_{Z_1} is also homeomorphic to an m-ball, so we may repeat this procedure, writing Z_1 as the union of closed intervals Z_1', Z_1'' of half the length of Z_1. We may then choose Z_2 to be one of these two intervals so that c remains non-zero in $S^n - B_{Z_2}$. Continuing inductively, we obtain a nested sequence of closed intervals

$$Z_1 \supset Z_2 \supset Z_3 \supset \dots$$

with the length of $Z_i = 2^{-i}$ and the homology class remains non-zero in $S^n - B_{Z_i}$. Let $z = \cap Z_i$ and note by induction that c must be 0 in $S^n - B_z$. Now the compact support axiom proposition 16.4, ensures that there is some compact subset $Y \subset S^n - B_z$ such that c is already 0 in Y. Compactness ensures that $Y \subset S^n - B_{Z_i}$ for i large enough so c is 0 in $S^n - B_{Z_i}$, a contradiction. This proves the lemma. □

Generalised Jordan curve theorem

Theorem 24.3 *Let $f : S^m \hookrightarrow S^n$ be an imbedding. Then*

$$\tilde{\mathrm{H}}_p(S^n - f(S^m)) = \begin{cases} 0 & \text{if } p \neq n - m - 1 \\ \mathbb{Z} & \text{if } p = n - m - 1 \end{cases}$$

Remark Note this gives the classical version theorem 24.1.

Proof. We argue by induction on m. If $m = 0$ then $f(S^0)$ is just a two point subset of S^n so $S^n - f(S^0) \simeq \mathbb{R}^n -$pt which is homotopy equivalent to S^{n-1} (if we define $S^{-1} = \varnothing$). But $\tilde{\mathrm{H}}_p(S^{n-1}) = 0$ unless $p = n - 1$ when the reduced homology is $\mathbb{Z}$. The theorem follows in this case.

For $m > 0$, we let E_+, E_- be the upper and lower hemispheres of S^n so $S^m = E_+ \cup E_-$ and the equator $E_+ \cap E_-$ may be identified with S^{m-1}. We examine the Mayer-Vietoris sequence of the open cover $\{X_+ = S^n - f(E_+), X_- = S^n - f(E_-)\}$ of $X = S^n - f(S^{m-1})$

$$H_{p+1}(X_+) \oplus H_{p+1}(X_-) \longrightarrow H_{p+1}(X) \xrightarrow{\partial} H_p(A) \longrightarrow H_p(X_+) \oplus H_p(X_-)$$

where $A = X_+ \cap X_- = S^n - f(S^m)$. The lemma 24.2 shows that the outer groups are 0 so ∂ above is an isomorphism. It follows by induction that

$$\tilde{\mathrm{H}}_p(A) \simeq \tilde{\mathrm{H}}_{p+1}(X) = \begin{cases} \mathbb{Z} & \text{if } p + 1 = n - (m-1) - 1 \\ 0 & \text{else} \end{cases}$$

This proves the theorem. □

Brouwer's invariance of domain theorem

Theorem 24.4 *Let $U \subseteq \mathbb{R}^n$ be an open subset and $f : U \longrightarrow \mathbb{R}^n$ be a continuous injective map. Then $f(U)$ is open and f is an imbedding.*

Remark: If f is continuously differentiable with non-singular Jacobian, then this follows from the inverse function theorem.

Proof. We may imbed $\mathbb{R}^n$ in S^n and consider the continuous injective map $f : U \longrightarrow S^n$.

It suffices to show that $f(U)$ is open, for then applying this to open subsets of U, we see that f is also open. To this end, we need only show that for any $u \in U$, we have $f(u)$ lies in the interior of $f(U)$.

Let $B_\varepsilon \subset U$ be a closed ε-ball centred at u and S_ε be its boundary ($\simeq S^{n-1}$). Theorem 24.3 shows that $\mathbb{R}^n - f(S_\varepsilon)$ has 2 connected components. Let V_{in} be the one containing $f(u)$ and V_{out} be the other. Since V_{in} is open, it suffices to show that $f(\mathrm{Int} B_\varepsilon) = V_{in}$.

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Now $f(\text{Int}B_\varepsilon)$ is connected and contains $f(u)$, so it must lie in V_{in}. If it is a proper subset, then

$$S^n - f(B_\varepsilon) = V_{out} \cup (V_{in} - f(\text{Int}B_\varepsilon))$$

is disconnected. This contradicts the acyclicity of $S^n - f(B_\varepsilon)$ 24.2. □

25 Cohomology

From this lecture on, we will not include full proofs. The proofs are no more difficult than the ones you have seen so far and use similar techniques. Details can be found in standard texts such as Munkres or Spanier.

Aim lecture We introduce a variant of homology called cohomology in the chain complex, simplicial and singular setting.

$$\mathbb{F} = \text{ field }, \quad \underline{\text{Ch}_\mathbb{F}} = \text{ category of chain complexes of } \mathbb{F}-\text{spaces}$$

Remark: We will only look at cohomology with co-efficients over a field $\mathbb{F}$. Similar versions hold with co-efficients over $\mathbb{Z}$ and you should be able to guess the definition if your module theory is good.

Contravariant functors

Let $\mathcal{C}, \mathcal{D}$ be categories.

Definition 25.1 *A* contravariant functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ *consists of the data:*

i. A function $Obj\mathcal{C} \longrightarrow Obj\mathcal{D} : X \mapsto F(X)$ *and,*

ii. for each $X, Y \in \mathcal{C}$ *a function*

$$F : Hom_\mathcal{C}(X, Y) \longrightarrow Hom_\mathcal{D}(F(Y), F(X))$$

such that

i. $F(id_X) = id_{F(X)}$ *and,*

ii. $F(fg) = F(g)F(f) : F(Z) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(X)$ *given* $X \xrightarrow{g} Y \xrightarrow{f} Z$ *in* $\mathcal{C}$*.*

E.g. (Linear duality functor) We define a contravariant functor $(-)^* : \underline{\text{Vect}_\mathbb{F}} \longrightarrow \underline{\text{Vect}_\mathbb{F}}$ on objects by the linear dual $V^* = \text{Hom}_\mathbb{F}(V, \mathbb{F})$ and on linear maps $f : V \longrightarrow W$ by $f^* : W^* \longrightarrow V^* : l \mapsto lf$. It is functorial since $\text{id}^* = \text{id}$ and

$$(fg)^*l = lfg = g^*(lf) = g^*f^*l.$$

Proposition 25.2 *The composite of a contravariant functor with a covariant functor is contravariant.*

Proof. Good ex to properly state and prove this. □

Facts about duality

We need some linear algebra facts concerning duality.

Lemma 25.3 *Let* $f : W \longrightarrow V$ *be a linear map. Then*

i. $\ker f^* = (im\ f)^\perp := \{l \in V^* | l(im\ f) = 0\}$.

ii. $im\ f^* = (\ker f)^\perp := \{l \in W^* | l(\ker f) = 0\}$.

Proof. For (i), note that $l \in V^*$ lies in $\ker f^*$ iff for all $v \in V$ we have

$$0 = (f^* l)v = lf(v)$$

and (i) follows.

For (ii), note $(f^*l)(\ker f) = l(f(\ker f)) = 0$ so certainly $\mathrm{im}\ f^* \subseteq (\ker f)^\perp$. Conversely, given $l \in W^*$ with $l(\ker f) = 0$, the universal property of quotients shows that there is some linear map $\bar{l} : W/\ker f \longrightarrow \mathbb{F}$ such that l is the composite $W \longrightarrow W/\ker f \xrightarrow{\bar{l}} \mathbb{F}$. Define $\tilde{l} : V \longrightarrow \mathbb{F}$ to be 0 on some vector space complement to $\mathrm{im}\ f < V$ and the composite $\mathrm{im}\ f \xrightarrow{\phi} W/\ker f \xrightarrow{\bar{l}} \mathbb{F}$ on $\mathrm{im}\ f$, where ϕ is the isomorphism in the first isomorphism theorem. Then $f^*\tilde{l} = l$ and (ii) holds. □

Corollary 25.4 *Let W be a subspace of V and consider the canonical SES*

$$0 \longrightarrow W \xrightarrow{\iota} V \xrightarrow{\pi} V/W \longrightarrow 0.$$

Then the dual sequence

$$0 \longrightarrow (V/W)^* \xrightarrow{\pi^*} V^* \xrightarrow{\iota^*} W^* \longrightarrow 0$$

is also exact. In this case, we can and will identify $W^\perp = (V/W)^*$.

Proof. The universal property of quotients can (ex) be restated as $\ker \iota^* = (V/W)^*$. It thus remains only to show that ι^* is surjective which follows as Lemma 25.3 ensures $\mathrm{im}\ \iota^* = (\ker \iota)^\perp = 0^\perp = W^*$. □

Dual chain complex

We now construct a contravariant duality functor $(-)^\vee : \underline{\mathrm{Ch}}_{\mathbb{F}} \longrightarrow \underline{\mathrm{Ch}}_{\mathbb{F}}$. Let $C_\bullet$ be a chain complex. We define a chain complex $C_\bullet^\vee$ by $C_p^\vee = C_{-p}^*$ and the boundary map $\partial^\vee$ to be

$$\partial_p^\vee = \partial_{-(p-1)}^* : C_p^\vee = C_{-p}^* \longrightarrow C_{p-1}^\vee = C_{-(p-1)}^*$$

Note this is a chain complex since $\partial_{p-1}^\vee \partial_p^\vee = \partial_{2-p}^* \partial_{1-p}^* = (\partial_{1-p}\partial_{2-p})^* = 0^* = 0$.

We define $(-)^\vee$ on chain maps as follows. For a chain map $f_\bullet : C_\bullet \longrightarrow D_\bullet$ we define $f_p^\vee = f_{-p}^* : D_p^* \longrightarrow C_p^*$. By functoriality of $(-)^*$ (ex), $f_\bullet^\vee$ is a chain map and functoriality of $(-)^\vee$ now follows from functoriality of $(-)^*$.

Definition 25.5 *We define the* p-th cohomology functor with co-efficients in $\mathbb{F}$, *denoted $H^p(-, \mathbb{F}) : \underline{Ch}_{\mathbb{F}} \longrightarrow \underline{Vect}_{\mathbb{F}}$ to be the composite contravariant functor*

$$\underline{Ch}_{\mathbb{F}} \xrightarrow{(-)^\vee} \underline{Ch}_{\mathbb{F}} \xrightarrow{H_{-p}} \underline{Vect}_{\mathbb{F}}.$$

The p-th cohomology group *of $C_\bullet$ with co-efficients in $\mathbb{F}$ is $H^p(C_\bullet, \mathbb{F})$.*

Warning This is not the usual definition!

Proposition 25.6 *There is a natural isomorphism $H^p(C_\bullet; \mathbb{F}) = H_p(C_\bullet; \mathbb{F})^*$.*

Proof.

$$C_{-p-1}^* \xrightarrow{\partial_{-p}^*} C_{-p}^* \xrightarrow{\partial_{1-p}^*} C_{-p+1}^*$$

Unravelling definitions shows that $H^{-p}(C_\bullet; \mathbb{F}) = (\ker \partial_{1-p}^*)/(\mathrm{im}\ \partial_{-p}^*)$. By lemma 25.3 and corollary 25.4, this is $(C_{-p}/\mathrm{im}\ \partial_{1-p})^*/(C_{-p}/\ker \partial_{-p})^*$. Dualising the injection $\ker \partial_{-p}/\mathrm{im}\ \partial_{1-p} \hookrightarrow C_{-p}/\mathrm{im}\ \partial_{1-p}$ yields a surjection

$$\left(\frac{C_{-p}}{\mathrm{im}\ \partial_{1-p}}\right)^* \longrightarrow \left(\frac{\ker \partial_{-p}}{\mathrm{im}\ \partial_{1-p}}\right)^* = H_{-p}(C_\bullet; \mathbb{F})^*$$

whose kernel is $(C_{-p}/\ker \partial_{-p})^*$. The first isomorphism theorem now shows $H^{-p}(C_\bullet;\mathbb{F}) = H_{-p}(C_\bullet;\mathbb{F})^*$. Checking naturality is long but easy and left to the reader. $\square$

Singular and simplicial cohomology

Given a simplicial complex K, we define the *p-th cohomology group* of K with co-efficients in $\mathbb{F}$ to be $H^p(C_\bullet(K);\mathbb{F})$. Given a topological space X, we define the *p-th cohomology group* of X with co-efficients in $\mathbb{F}$ to be $H^p(S_\bullet(X);\mathbb{F})$. Composing $H^p : \underline{\mathrm{Ch}_{\mathbb{F}}} \longrightarrow \underline{\mathrm{Vect}_{\mathbb{F}}}$ with the functors $C_\bullet(-;\mathbb{F}) : \underline{\mathrm{Simp}} \longrightarrow \underline{\mathrm{Ch}_{\mathbb{F}}}$, $S_\bullet(-;\mathbb{F}) : \underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Ch}_{\mathbb{F}}}$ give contravariant functors

$$H^p(-;\mathbb{F}) : \underline{\mathrm{Simp}} \longrightarrow \underline{\mathrm{Vect}_{\mathbb{F}}}, \quad H^p(-;\mathbb{F}) : \underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Vect}_{\mathbb{F}}}.$$

26 Cup products

Aim lecture Show that cohomology has the advantage over homology because it has an extra ring structure.

Remark Homology has a coring structure, which is less intuitive than ring structures.

$$\mathbb{F} = \text{ field }, \quad X = \text{ topological space}, \quad K = \text{ simplicial complex}$$

$\mathbb{F}$-algebras

An $\mathbb{F}$-*algebra* is a ring A which has a compatible structure of a vector space over $\mathbb{F}$. By a compatible structure we mean that the addition in both is the same and ring multiplication is $\mathbb{F}$-bilinear. Since the distributive law holds by ring axioms, this just means for $\beta \in \mathbb{F}, a, a' \in A$, that $(\beta a)a' = \beta(aa') = a(\beta a')$. For an $\mathbb{F}$-algebra A, we may identify $\mathbb{F}$ with the subring $\mathbb{F}\,1$. An *algebra homomorphism* is just an $\mathbb{F}$-linear ring homomorphism. Let $\underline{\mathrm{Alg}_{\mathbb{F}}}$ denote the category of $\mathbb{F}$-algebras and algebra homomorphisms.

Proposition 26.1 *Let V, W be subspaces of an $\mathbb{F}$-algebra A. If VW denotes that subspace spanned by all products $vw, v \in V, w \in W$, then* $\dim VW \leq (\dim V)(\dim W)$.

Proof. If $\{v_i\} \subset V, \{w_j\} \subset W$ are bases, then $\{v_i w_j\}$ spans VW by bilinearity of ring multiplication. $\square$

Cohomology ring

We let

$$H^*(X;\mathbb{F}) = \oplus_p H^p(X;\mathbb{F}), \quad H^*(K;\mathbb{F}) = \oplus_p H^p(K;\mathbb{F}).$$

Theorem 26.2 *There exists multiplication maps $\cup$ on $H^*(X;\mathbb{F}), H^*(K;\mathbb{F})$ which make them $\mathbb{F}$-algebras. Furthermore, the cohomology functors H^p sum to give contravariant functors*

$$H^* : \underline{HTop} \longrightarrow \underline{Alg_{\mathbb{F}}} : X \mapsto H^*(X;\mathbb{F}),\ (X \xrightarrow{f} Y) \mapsto \oplus_p H^p(f) =: f^*$$

$$H^* : \underline{Simp} \longrightarrow \underline{Alg_{\mathbb{F}}} : K \mapsto H^*(K;\mathbb{F}),\ (K \xrightarrow{g} L) \mapsto \oplus_p H^p(g) =: g^*$$

The multiplication is graded in the sense that $H^p(X;\mathbb{F}) \cup H^q(X;\mathbb{F}) \subseteq H^{p+q}(X;\mathbb{F})$

Proof. Omitted. We will however, indicate how $\cup$, called the *cup product* is defined. By the distributive law, we need only define maps

$$\cup : H^p(X;\mathbb{F}) \times H^q(X;\mathbb{F}) \longrightarrow H^{p+q}(X;\mathbb{F}).$$

This will be induced by another cup product map

$$\cup : S_p(X;\mathbb{F})^* \times S_q(X;\mathbb{F})^* \longrightarrow S_{p+q}(X;\mathbb{F})^*$$

defined as follows. Consider the "front p-face" affine linear map $f^p = l(\Delta_p, \langle\, \varepsilon_0 \ldots \varepsilon_p\, \rangle) : \Delta_p \longrightarrow \Delta_{p+q}$ and "back q-face affine linear map $b^q = l(\Delta_p, \langle\, \varepsilon_p \ldots \varepsilon_{p+q}\, \rangle) : \Delta_p \longrightarrow \Delta_{p+q}$. Given $c^p \in S_p(X;\mathbb{F})^*, c^q \in S_q(X;\mathbb{F})^*$ we define the linear map $c^p \cup c^q : S_{p+q}(X;\mathbb{F}) \longrightarrow \mathbb{F}$ on a singular $p+q$-simplex $T : \Delta_{p+q} \longrightarrow X$ to be

$$(c^p \cup c^q)(T) = c^p(Tf^p)c^q(Tb^q).$$

One can show it induces a multiplication map on cohomology because of the following "Leibniz formula"

$$\partial^*(c^p \cup c^q) = \partial^* c^p \cup c^q + (-1)^p c^p \cup \partial^* c^q.$$

The simplicial case is similar. □

Remark The grading ensures that the unit $1 \in H^0(X;\mathbb{F})$.

Example 26.3 $H^*(S^1;\mathbb{F}) \simeq \mathbb{F}[x]/(x^2)$.

Proof. We know that $H_p(S^1;\mathbb{F}) \simeq \mathbb{F}$ if $p = 0, 1$ and is 0 otherwise. Hence by proposition 25.6, the (graded) vector spaces $H^*(S^1;\mathbb{F}) = H^0(S^1;\mathbb{F}) \oplus H^1(S^1;\mathbb{F}) \simeq \mathbb{F}\,1 \oplus \mathbb{F}\,u$ where u is any non-zero element of $H^1(S^1;\mathbb{F})$. Now $u^2 \in H^2(S^1;\mathbb{F}) = 0$. Hence there is an $\mathbb{F}$-algebra homomorphism $\mathbb{F}[x]/(x^2) \longrightarrow H^*(S^1;\mathbb{F}) : x \mapsto u$ which must be bijective as it's surjective and both domain and co-domain are 2-dimensional. □

The cohomology ring is "super-commutative" in the following sense.

Proposition 26.4 *Given $c^p \in H^p(X;\mathbb{F}), c^q \in H^q(X;\mathbb{F})$ we have*

$$c^q \cup c^p = (-1)^{pq} c^p \cup c^q.$$

Proof. Omitted. □

Example 26.5 *Let $c \in H^p(X;\mathbb{F})$ where p is odd and char $\mathbb{F} \neq 2$. Then $c \cup c = 0$.*

Proof. Since $c \cup c = -c \cup c$. □

Künneth formula

Consider the product space $X = X_1 \times X_2$ and let $\pi_i : X \longrightarrow X_i$ be the two projections. We have two algebra homomorphisms $\pi_i^* : H^*(X_i;\mathbb{F}) \longrightarrow H^*(X;\mathbb{F})$. The following Künneth formula computes $H^*(X;\mathbb{F})$ in terms of $H^*(X_1;\mathbb{F})$ and $H^*(X_2;\mathbb{F})$.

Theorem 26.6 *i.* *π_i^* is injective.*

ii. *$H^n(X;\mathbb{F})$ is the internal direct sum of the subspaces*

$$V_{p,n-p} := \pi_1^*\left(H^p(X_1;\mathbb{F})\right) \cup \pi_2^*\left(H^{n-p}(X_2;\mathbb{F})\right)$$

for $p = 0, 1, \ldots n$.

iii. $\dim V_{p,n-p} = (\dim(H^p(X_1;\mathbb{F}))(\dim(H^{n-p}(X_2;\mathbb{F}))$.

Proof. Omitted. ☐

Example 26.7 *Cohomology of* $\mathbb{T}^2 \simeq S^1 \times S^1$.

Let u be a basis for $H^1(S^1;\mathbb{F})$ and

$$u_1 = \pi_1^* u = \pi_1^* u \cup 1, \quad u_2 = \pi_2^* u \in H^1(\mathbb{T}^2;\mathbb{F}).$$

The Künneth formula tell us that

$$\begin{aligned}
H^0(\mathbb{T}^2;\mathbb{F}) &= \pi_1^* H^0(S^1;\mathbb{F}) \cup \pi_2^* H^0(S^1;\mathbb{F}) = \mathbb{F} \\
H^1(\mathbb{T}^2;\mathbb{F}) &= \left(\pi_1^* H^0(S^1;\mathbb{F}) \cup \pi_2^* H^1(S^1;\mathbb{F})\right) \oplus \left(\pi_1^* H^1(S^1;\mathbb{F}) \cup \pi_2^* H^0(S^1;\mathbb{F})\right) = \mathbb{F}\, u_2 \oplus \mathbb{F}\, u_1 \\
H^2(\mathbb{T}^2;\mathbb{F}) &= \pi_1^* H^1(S^1;\mathbb{F}) \cup \pi_2^* H^1(S^1;\mathbb{F}) = \mathbb{F}\, u_1 \cup u_2
\end{aligned}$$

Note the multiplication is completely determined e.g.

$$u_1 \cup u_1 = \pi_1^* u \cup \pi_1^* u = \pi_1^*(u \cup u) = 0.$$

In fact, the Künneth formula, super commutativity, and the fact that π^* are homomorphisms, give a complete description of the cohomology ring $H^*(X_1 \times X_2;\mathbb{F})$ in terms of the $H^*(X_i;\mathbb{F})$. If you know about tensor products, the fancy way of expressing this is

Theorem 26.8 $H^*(X_1 \times X_2;\mathbb{F}) \simeq H^*(X_1;\mathbb{F}) \otimes_{\mathbb{F}} H^*(X_2;\mathbb{F})$ *where the algebra structure is as the tensor product of super algebras.*

27 Poincaré duality and applications

Aim lecture We give the Poincaré duality theorem which gives extra structure on the cup product in the case of manifolds. We give applications to the real projective plane and some topological applications.

$$\mathbb{F} = \text{field}$$

Poincaré duality

A *compact topological n-manifold* is a compact Hausdorff space X such that every point $x \in X$ has an open neighbourhood homeomorphic to $\mathbb{R}^n$. If X is also connected, we say it is *orientable* if $H_n(X) \neq 0$.

Theorem 27.1 *Let X be a connected, compact topological n-manifold which is* triangulable *i.e. there exists a triangulation. Let $\mathbb{F}$ = arbitrary field if X is orientable, and $\mathbb{F}_2$ if not. Then $H^n(X;\mathbb{F}) \simeq \mathbb{F}$ and for any $0 \leq p \leq n$ and non-zero $c \in H^p(X;\mathbb{F})$, the linear map $c\cup : H^{n-p}(X;\mathbb{F}) \longrightarrow H^n(X;\mathbb{F}) : c' \mapsto c \cup c'$ is non-zero.*

Proof. Omitted. We only remark that the theorem is called Poincaré duality because it implies that $H^p(X;\mathbb{F})$ and $H^{n-p}(X;\mathbb{F})$ are naturally dual vector spaces via the cup product. ☐

Cohomology ring of $\mathbb{R}P^2$

Definition 27.2 *Let $\sim$ be the weakest equivalence relation on $\mathbb{R}^n$ (or subset of $\mathbb{R}^n$) defined by $x \sim -x$ for all $x \in X$. We define the* n-dimensional real projective space *to be the quotient space $\mathbb{R}P^n := S^n/\sim$. It is a compact connect triangulable topological n-manifold.*

Ex The *real projective line* $\mathbb{R}P^1$ is homeomorphic to S^1 whilst $\mathbb{R}P^2$ is homeomorphic to the real projective plane defined in lecture 3.

Proposition 27.3 *$H^*(\mathbb{R}P^2;\mathbb{F}_2) \simeq \mathbb{F}_2[x]/(x^3)$.*

Proof. By picking a triangulation of $\mathbb{R}P^2$, one can easily compute as in lecture 5 the homology of $\mathbb{R}P^2$ as $H_p(\mathbb{R}P^2;\mathbb{F}_2) \simeq \mathbb{F}_2$ if $p = 0, 1, 2$ and is 0 otherwise. We conclude from proposition 25.6, that $H^p(\mathbb{R}P^2;\mathbb{F}_2) \simeq \mathbb{F}_2$ if $p = 0, 1, 2$ and is 0 otherwise.

It remains only to determine the product structure, so let u be a basis for $H^1(\mathbb{R}P^2;\mathbb{F}_2)$. Note $u^3 \in H^3(\mathbb{R}P^2;\mathbb{F}_2) = 0$, so there is an algebra homomorphism $\phi : \mathbb{F}_2[x]/(x^3) \longrightarrow H^*(\mathbb{R}P^2;\mathbb{F}_2)$ which maps $x \mapsto u$. Poincaré duality thm 27.1 tells us that $u\cup : \mathbb{F}_2\, u \longrightarrow H^2(\mathbb{R}P^2;\mathbb{F}_2)$ is non-zero so $u \cup u$ must be a non-zero and hence basis element for $H^2(\mathbb{R}P^2;\mathbb{F}_2)$. It follows that ϕ is surjective and counting dimensions, bijective. □

Corollary 27.4 *Let $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^1$ be continuous. Then $H_1(f;\mathbb{F}_2) = 0$.*

Proof. We consider the algebra homomorphism

$$f^* : H^*(\mathbb{R}P^1;\mathbb{F}_2) \simeq \mathbb{F}_2[y]/(y^2) \longrightarrow H^*(\mathbb{R}P^2;\mathbb{F}_2) \simeq \mathbb{F}_2[x]/(x^3)$$

where $x, y \in H^1$. If $f^*y = ax, a \in \mathbb{F}_2$ then $0 = f^*y^2 = a^2x^2$ so $a = 0$ and $H^1(f;\mathbb{F}_2) = 0$. Hence by proposition 25.6, $H_1(f;\mathbb{F}_2) = 0$. □

Baby Borsuk-Ulam theorem and the ham sandwich

Lemma 27.5 *Let $q : S^1 \longrightarrow \mathbb{R}P^1 = S^1/\sim$ be the natural degree two quotient map. If $T : I \longrightarrow S^1$ is any path from $x \in S^1$ to $-x$, then $q_\# T \in H_1(\mathbb{R}P^1;\mathbb{F}_2)$ is a cycle representing a non-zero homology class.*

Proof. Let $T' : I \longrightarrow S^1$ be another path from x to $-x$. We first show that $q_\# T = q_\# T'$. Indeed, $T - T'$ is a 1-cycle and q has degree two so as 2 annihilates $\mathbb{F}_2$ we have $q_\#(T - T') = 0$ as desired. We may thus assume that $T : I \longrightarrow S^1$ is the path defined by $T(t) = (\cos \pi t, \sin \pi t)$. The result now follows from theorem 23.2. □

We say a continuous map $f : S^n \longrightarrow S^m$ is *antipode preserving* if $f(-x) = -f(x)$ for all $x \in S^n$.

Lemma 27.6 *There is no antipode preserving map $f : S^2 \longrightarrow S^1$.*

Proof. Suppose f were an antipode preserving map. Then there is an induced commutative diagram in Top

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & S^1 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{R}P^2 & \xrightarrow{g} & \mathbb{R}P^1 \end{array}$$

We will derive a contradiction to corollary 27.4 by showing that $H_1(g;\mathbb{F}_2) \neq 0$. Let $T : I \longrightarrow S^2$ be any path from some point x to $-x$. Then since f is antipode preserving, we may apply lemma 27.5 to $f_\# T$ to see that $q_\# T$ is a cycle of $\mathbb{R}P^2$ such that $g_\# q_\# T = q_\# f_\# T \neq 0$. The lemma follows. □

We can now prove a baby version of the Borsuk-Ulam theorem.

Theorem 27.7 *Let $f : S^2 \longrightarrow \mathbb{R}^2$ be a continuous map. Then there is some $x \in S^2$ such that $f(-x) = f(x)$.*

Proof. If the theorem were false, then we would obtain a continuous function

$$g : S^2 \longrightarrow S^1 : x \mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

This is antipode preserving so contradicts lemma 27.6. □

Theorem 27.8 *Let $A_1, A_2 \subset \mathbb{R}^2$ be two bounded measurable subsets. There is a line $L \subset \mathbb{R}^2$ that divides both A_1, A_2 into equal areas.*

Remark If A_1 denotes a slice of bread, and A_2 a slice of ham, then one can cut both of them in half with a single cut.

Proof. $\mathbb{R}P^2$ naturally enters the proof as its points parametrise the lines in $\mathbb{R}^2$ plus the line at ∞ whilst its double cover S^2 parametrises these lines and a choice of side of the line. To see this, it is convenient to place the A_i in the $z = 1$ plane in xyz-space. If μ denotes the Lebesgue measure on $\mathbb{R}^2$ we let

$$\mu_i(\vec{u}) = \mu\{\vec{v} \in A_i | \vec{u} \cdot \vec{v} \geq 0\}$$

for $\vec{u} \in S^2, i = 1, 2$. Then $\mu_i(\vec{u}) + \mu_i(-\vec{u}) = \mu(A_i)$. We apply baby Borsuk-Ulam to the map $f : S^2 \longrightarrow \mathbb{R}^2 : \vec{u} \mapsto (\mu_1(\vec{u}), \mu_2(\vec{u}))$ to find $\vec{w} \in S^2$ with $f(\vec{w}) = f(-\vec{w})$. The line in the $z = 1$ plane orthogonal to $\vec{w}$ works. □

28 Cohomology rings of surfaces

In this lecture, X is a surface, by which we mean a connected compact orientable topological 2-manifold which is triangulable.

Symplectic forms

Let V = fin dim $\mathbb{R}$-space and $(-,-) : V \times V \longrightarrow \mathbb{R}$ be a bilinear form on V i.e. for all $v \in v, (v,-)$ and $(-,v)$ are linear. We say that $(-,-)$ is *symplectic* if for all $v \in V$ we have $(v,v) = 0$ but $(v,-) \neq 0$. In particular, we have $(v,w) = -(w,v)$.

Example 28.1 *The hyperbolic plane $\mathbb{H}$.*

Let $\mathbb{H} = \mathbb{R}^2$ and $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Consider the bilinear form $(v,w) = v^T A w$. This is a symplectic form on $\mathbb{H}$ since A is skew-symmetric and A is invertible. $\mathbb{H}$ with this form is called the *hyperbolic plane*.

Example 28.2 $V = H^1(X;\mathbb{R})$.

By Poincaré duality there is an isomorphism $\phi : H^2(X;\mathbb{R}) \xrightarrow{\sim} \mathbb{R}$ and also a symplectic form on $V = H^1(X;\mathbb{R})$ defined by $(v,w) = \phi(v \cup w)$.

Proposition 28.3 *Let V be an $\mathbb{R}$-space with a symplectic form $(-,-)$. Then V is the orthogonal direct sum of hyperbolic planes. In other words, it has a basis ...*

Proof. Induction ☐

Corollary 28.4 *The cohomology ring of a surface is*

In particular, $H^1(X;\mathbb{R})$ is even dimensional, say of dim $2g$ where g is the *genus* of X. The Euler number can then be written as $e(X) = 2 - 2g$.

Connected sums

Recall given two surface X, Y, the *connected sum* $X\#Y$ is obtained by removing a small disc from each and gluing the remnants along the common circular boundary

DRAW PICTURE

It is still a surface.

Fact Every surface is homeomorphic to the connected sum $X_g := \mathbb{T}^2\#\ldots\#\mathbb{T}^2$ of g copies of $\mathbb{T}^2$ with itself.

We won't prove this which is usually proven in MATH3701. The following however shows that the genus of X_g is g so these surfaces are non-homeomorphic.

Proposition 28.5 *For surfaces X, Y, $e(X\#Y) = e(X) + e(Y) - 2$.*

Why? Easiest to see from simplicial homology.

Additivity of Euler characteristic

Alternatively, we can use the additivity of Euler characteristic which is easy to prove in the case of an admissible cover.

Proposition 28.6 *Let $\mathcal{A} = \{Y_1, Y_2\}$ be an admissible cover of a topological space Y and $A = Y_1 \cap Y_2$. Then $e(Y) = e(Y_1) + e(Y_2) - e(A)$.*

Proof. We use the M-V sequence and the next lemma. ☐

Lemma 28.7 *Consider a SES of complexes $0 \longrightarrow C_\bullet \longrightarrow D_\bullet \longrightarrow E_\bullet \longrightarrow 0$. Then*

$$\sum_p (-1)^p \dim H_p(D_\bullet) = \sum_p (-1)^p \dim H_p(C_\bullet) + \sum_p (-1)^p \dim H_p(E_\bullet).$$

Proof. Let ∂ be the connecting homomorphism in the LES in homology. We use the SESs

$$0 \longrightarrow H_p(C)/\operatorname{im}\partial_{p+1} \longrightarrow H_p(D) \longrightarrow \ker\partial_p \longrightarrow 0$$

$$0 \longrightarrow \ker\partial_p \longrightarrow H_p(E) \longrightarrow \operatorname{im}\partial_p \longrightarrow 0$$

☐

29 De Rham cohomology

Aim Lecture Give a different version of cohomology with co-efficients in $\mathbb{R}$ in the special case of manifolds.

Review differential forms

Let $U \subseteq \mathbb{R}^n_{x_1,\ldots,x_n}$ be an open set, $C^\infty(U)$ be the space of infinitely differentiable functions. The space of *differential p-forms* on U is

$$E^p = \{ \sum_{i_1<\ldots<i_p} f_{i_1\ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p} | f_{i_1\ldots i_p} \in C^\infty(U)\}$$

Here $E^0 = C^\infty$. Recall $E^*(U)$ is a ring with the multiplication given by $\wedge$ and the rule

$$dx_i \wedge dx_i = 0, \quad dx_i \wedge dx_j = -dx_j \wedge dx_i$$

We can also define the exterior derivative $d : E^p(U) \longrightarrow E^{p+1}(U)$ by

$$d\left(\sum_{i_1<\ldots<i_p} f_{i_1\ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right) = \sum_{i_1<\ldots<i_p} \frac{\partial f_{i_1\ldots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

Example 29.1 *Gradient and curl in* $\mathbb{R}^2$*.*

An interpretation of kernel mod image

Often (co)homology arises naturally in the following context. The kernel is the set of solutions of some equation of interest e.g. $\operatorname{curl}\mathbf{F} = 0$ whose solutions are the irrotational vector fields, say in an open set $U \subseteq R^2$. We may be able to construct lots of "trivial" solutions which one can often express as the image of some map. In our example, these might be the image of the gradient map, i.e. gradients of scalar potential functions. Then $H^1 = \ker \operatorname{curl} / \operatorname{im} \operatorname{grad}$ is the set of solutions modulo "trivial solutions".

If U is simply connected, we know that $H^1 = 0$. However, we have

Example 29.2 $\mathbf{F}(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ *is irrotational but not the gradient of a potential function.*

This "captures" the non-acyclicity of $\mathbb{R}^2 - (0, 0)$.

De Rham cohomology

Let X be an oriented n-dim manifold. This means in particular, that every point $x \in X$ has at least one neighbourhood with a given homeomorphism to an open set in $\mathbb{R}^n$. Furthermore, all such homeomorphisms are compatible so that we can use them to define the space of (global) differential p-forms $E^p(X)$ independent of co-ordinates. We thus also obtain $E^*(X)$ and a de Rham complex

$$C_\bullet^{DR} : 0 \longrightarrow E^0(X) \xrightarrow{d} E^1(X) \xrightarrow{d} E^2(X) \longrightarrow \dots$$

with $C_p^{DR} = E^{-p}(X)$ and boundary operator d is the exterior derivative. We define the *p-th de Rham cohomology* of X to be

$$H^p_{DR}(X) := H_{-p}(C_\bullet^{DR}(X)).$$

Theorem 29.3 *$H^p_{DR}(X) \simeq H^p(X; \mathbb{R})$.*

It is easy to define the isomorphism $\Phi : H^p_{DR}(X) \longrightarrow H_p(X;\mathbb{R})^*$. Let $T : \Delta_p \longrightarrow X$ be a singular p-simplex that is furthermore differentiable and $\omega \in E^p(X)$. Then $\Phi(\omega)$ maps

$$T \mapsto \int_{\Delta_p} T^*\omega.$$

Stokes's thm $\int_{\partial c} \omega = \int_c d\omega$ ensures this descends to the (co)homology level. It turns out that for manifolds, cohomology over $\mathbb{R}$ can be computed using the chain complex of differentiable singular simplices so we're done.

Many features of cohomology can be seen alternatively from this viewpoint. For example, the cup product corresponds to the wedge product of forms and super-commutativity follows from super-commutativity of the wedge product.

Poincaré duality is also easy to understand.

Theorem 29.4 *Let X be a compact oriented n-dim manifold. Then*

$$\int_X : H^n_{DR}(X) \longrightarrow \mathbb{R} : \omega \mapsto \int_X \omega$$

is a well-defined isomorphism.

Example 29.5 *If $X = S^1$, then $H^p_{DR}(X) = \mathbb{R}$ if $p = 0, 1$ and 0 otherwise.*

We analyse the de Rham complex

$$0 \longrightarrow E^0(S^1) = C^\infty(S^1) \xrightarrow{d} E^1(S^1) \longrightarrow 0.$$

For $f \in C^\infty(S^1)$ we have $df = 0$ iff f is constant. Hence $H^0_{DR}(S^1) = \mathbb{R}$. For H^1, note that the "angle function" θ is a multi-valued function on S^1. Picking branches however, shows that $d\theta$ is an honest global differential 1-form. Note $E^1(S^1) = C^\infty(S^1)d\theta$. Now $d\theta$ is a 1-cycle and it's non-zero in cohomology because

$$\int_{S^1} d\theta = 2\pi$$

so $d\theta$ induces a non-zero linear functional on $H_1(S^1;\mathbb{R})$.

Hence to show $H^1_{DR}(S^1) = \mathbb{R}$, it suffices to show every 1-form $fd\theta$ is (co)homologous to $cd\theta$ for some $c \in \mathbb{R}$. To determine c, remember $fd\theta$ and $cd\theta$ should induce the same linear functional on H_1 i.e.

$$c = \frac{1}{2\pi}\int_{S^1} f d\theta$$

This ensures

$$g(\theta) := \int_0^\theta (f - c) d\theta$$

is a well-defined function on S^1 (as opposed to a multi-valued one). Calculus tells us

$$dg = fd\theta - cd\theta$$

so we're done.

30 General themes in algebraic topology

Aim lecture We recap some of the main ideas and themes in algebraic topology and indicate how they have been developed further.

X = topological space

Topological invariants

We have seen that a topological invariant is often given by a functor $F : \underline{\text{Top}} \longrightarrow \mathcal{C}$ or better still $\underline{\text{HTop}} \longrightarrow \mathcal{C}$ where $\mathcal{C}$ is a category consisting of algebraic objects such as $\mathcal{C} = \underline{\text{Ab}}, \underline{\text{Vect}_{\mathbb{F}}}$ etc. We've looked at several examples such as $H_p, H^p(-;\mathbb{F})$ etc. In the setting of pointed topological spaces, we also have the fundamental group functor $\pi_n : \underline{\text{PtTop}} \longrightarrow \underline{\text{Grp}}$. Coarser topological invariants for a topological space X are often given by considering invariants of algebraic objects e.g. the dimension of the homology vector spaces give Betti numbers etc.

Invariants given by functors $\underline{\text{HTop}} \longrightarrow \mathcal{C}$ are (by the simple axioms of a functor) well-adapted showing non-existence of various continuous maps and in particular, that certain spaces are not homeomorphic. Importantly, many of these non-existence results can yield existence results e.g. Brouwer's (existence of) fixed points theorem, the Fundamental thm of algebra, Ham sandwich thm etc.

Algebraic topology actually has a whole host of interesting functors $\underline{\text{Top}} \longrightarrow \mathcal{C}$ and finding more is an important task. Ideally, one would like enough invariants, that they alone can be used to distinguish between homotopy types of topological spaces.

One such other topological invariant is

K-theory

One can study a topological space by looking at all vector bundles on it. Let $\underline{\text{ComHaus}}$ be the category of compact Hausdorff spaces and continuous maps and $X \in \underline{\text{ComHaus}}$. A rank $r \in \mathbb{N}$ *(complex) vector bundle* V on X is essentially a continuous family of r-dimensional $\mathbb{C}$-spaces $\{V_x | x \in X\}$. If X is a smooth manifold, the tangent bundle T is one such example, which assigns to each point $x \in X$, its tangent space T_x. We may consider the set $K'(X)$ of isomorphism classes $[V]$ of vector bundles on X. Fibrewise direct sum of vector spaces gives a direct sum vector bundle on X. This induces an addition operation on $K'(X)$ i.e.

$$[V] + [W] := [V \oplus W].$$

Negatives don't exist, but we can formally introduce them, much as we do to natural numbers when constructing $\mathbb{Z}$, to define an abelian group $K(X)$.

Given a continuous map $f : X \longrightarrow Y$ and a vector bundle W on Y, there is a pull-back bundle $V = f^*W$ essentially defined by $V_x = W_{f(x)}$. It turns out that this induces a group homomorphism $f^* : K(Y) \longrightarrow K(X)$. Hence we obtain a functor $K : \underline{\text{ComHaus}} \longrightarrow \underline{\text{Ab}}$.

Actually, we can take fibrewise tensor products of vector bundles to define tensor products which gives a ring structure on $K(X)$ and hence we have a functor $K : \underline{\text{ComHaus}} \longrightarrow \underline{\text{Ring}}$ where $\underline{\text{Ring}}$ is the category of rings and ring homomorphisms.

Example 30.1 $X =$ *point*

$K(\text{pt}) \simeq \mathbb{Z}$. Why? The isomorphism is given by the dimension function.

Relationship between higher homotopy groups, homology and K-theory

The three invariants, homotopy groups, (co)homology and K-theory are not unrelated. This means on the one hand, that there is a lot of overlap of information, but on the other hand, you can often use one to obtain information about another.

The Hurewicz homomorphism relates π_n with H_n. It's an easy fact that $\pi_n(X, x)$ is abelian if $n \geq 2$, but not necessarily so if $n = 1$. Given a group G, its *abelianisation* G_{ab} is the maximal abelian quotient i.e. $G_{ab} = G/[G, G]$ where $[G, G]$ is the commutator subgroup generated by $g^{-1}h^{-1}gh$ for all $g, h \in G$. For an integer $n \geq 1$, we say X is $n-1$-*connected* if X is path-connected and $\pi_i(X, x) = 0$ for $1 \leq i < n$.

Theorem 30.2 (Hurewicz) *If X is $n-1$-connected then $\pi_n(X, x)_{ab} \simeq H_n(X, x) \simeq H_n(X)$.*

The isomorphism Φ is given as follows. Let $\alpha : (S^n, \mathrm{pt}) \longrightarrow (X, x)$ be a continuous map representing an element of $\pi_n(X, x)$. We fix a generator of $c \in H_n(S^n, \mathrm{pt}) \simeq \mathbb{Z}$. Then $\Phi([\alpha]) = \alpha_* c$.

Warning The thm says the abelianisation of the first non-zero homotopy group is isomorphic to the corresponding homology. But note $\pi_3(S^2) \simeq \mathbb{Z}$ and is in fact generated by the Hopf fibration $S^3 \hookrightarrow \mathbb{C}^2 - 0 \xrightarrow{q} \mathbb{C}P^1 \simeq S^2$.

For K-theory, there is a Chern character map $K(X) \longrightarrow H^*(X; \mathbb{Q})$, which is a ring homomorphism.

Computing (co)homology

We have seen two basic methods for calculating (co)homology: a) using the main theorem and simplicial homology which is purely combinatorial and b) using "formulas" for spaces built from others such as the Mayer-Vietoris sequence and Künneth formula.

The method of simplicial homology for polytopes can be extended to more general topological spaces called CW-complexes. These are formed by succesively building up a space by attaching balls B^n with n non-decreasing. e.g.

There is an analogue of the simplicial chain complex in this case, with one generator for each ball attached. Unfortunately, the boundary maps are harder to describe. Nevertheless, this is an important method for computing homology. In particular, the homology of projective spaces are easy to compute this way.

There are other homology "formulas" for example, for *fibre bundles* which are essentially spaces which are only locally products $X \times Y$ (say locally on Y). A vector bundle for example is locally a product of X with $\mathbb{C}^n$ so gives a fibre bundle, and it gives lots of other interesting ones too. The homology of a fibre bundle is described by an algebraic gadget, that is even more abstruse than a long exact sequence called a spectral sequence.

Alternate versions of (co)homology. Eilenberg-Steenrod axioms

We have seen that if X is a polytope, then we can calculate its homology using simplicial homology, whilst if it is a manifold, we can instead use de Rham cohomology to compute its (real) cohomology. We naturally asks, are their other ways to compute (co)homology. There are a whole host of them e.g. Cech cohomology is another which generalises nicely to cohomology of sheaves.

One way to check that they do indeed give the usual (co)homology is to check that they satisfy what's known as the Eilenberg-Steenrod axioms. Roughly speaking, this means, a)H_n (or H^n) form functors from $\underline{\text{HTopPair}}$ (or suitable subcategory) to $\underline{\text{Ab}}$ or $\underline{\text{Vect}_{\mathbb{F}}}$, b) there are natural LES for a pair (X, A), c) excision holds d) a point is acyclic. Anyway sequence of functors satisfying these give (co)homology.

Excision fails for homotopy groups.

Extra structure on (co)homology & K-theory

We have seen that cohomology is nicer than homology because it has an extra ring structure. It is natural to ask what extra structure we can put on (co)homology, or K-theory or any of the other topological invariants floating around.

A common one in algebraic topology is the notion of a *cohomology operation* which is a natural transformation $\eta : H^p(-;\mathbb{F}) \longrightarrow H^q(-;\mathbb{F})$ for some $p, q, \mathbb{F}$. Hence given a contiuous map $f : X \longrightarrow Y$, there's a commutative diagram.

$$\begin{array}{ccc} H^p(Y;\mathbb{F}) & \xrightarrow{\eta_Y} & H^q(Y;\mathbb{F}) \\ {\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow \\ H^p(X;\mathbb{F}) & \xrightarrow{\eta_X} & H^q(X;\mathbb{F}) \end{array}$$

One example in algebraic topology is the *i-th Steenrod square* which is a natural transformation $Sq^i : H^p(X;\mathbb{F}_2) \longrightarrow H^{p+i}(X;\mathbb{F}_2)$ for any p. One can show that $X = S(\mathbb{C}P^2)$ and the 1-point union Y of S^3 with S^5 have the same cohomology ring, but they are not homotopy equivalent because Sq^2 is trivial for one, but not the other.

In K-theory, operations are natural transformations $K \longrightarrow K$ and serve a similar function. Sending a vector bundle $V \mapsto V^{\otimes k}$ is one such, but not interesting as it is just the k-th power map in the ring. However, S_k acts on $V^{\otimes k}$ by permuting tensor factors, so we can perform a Fourier decomposition on $V^{\otimes k}$ to arrive at all sorts of associated vector bundles e.g. the symmetric power, the exterior power, and the Adams operations (which corresponds to the conjugacy class of k-cycles). The last can be used to show the following theorem: if $\mathbb{R}^n$ has a continuous bilinear multiplication map without zero divisors, then $n = 1, 2, 4$ or 8. Of course for these values, we have the (not necessarily associative) division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H} =$ the quaterions and $\mathbb{O} =$ the octonions. Details can be found in Steve Siu's thesis on my webpage.