Lecture 29: Weil Conjectures

Weil's Insight: Let V = proj. variety $/\mathbb{Z}$ smooth $/\mathbb{C}$.
Then $|V(\mathbb{F}_{p^s})|$ is profoundly determined by the topology of $V(\mathbb{C})$, for most primes p!

Here smooth $/\mathbb{C}$ means $V(\mathbb{C})$ is a complex manifold.

$\boxed{\text{Setup}}$ Fix p prime & l distinct prime.
Assume $V(\overline{\mathbb{F}}_p)$ smooth,
if $V = V(f(x_0, \dots, x_n))$ this means
$$V\left(f, \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)(\overline{\mathbb{F}}_p) = \emptyset.$$

Problem: Study $N_s = |V(\mathbb{F}_{p^s})|$ as a function of s. Equiv., study the zeta function

$$Z(V,t) = \exp\left(\sum N_s \frac{t^s}{s}\right)$$

Weil's Idea

Recall Frobenius homomorphism $Fr : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p : \alpha \mapsto \alpha^p$ is a ring hom. so get well-defined Frobenius endomorphism

$$\varphi : V(\overline{\mathbb{F}}_p) \longrightarrow V(\overline{\mathbb{F}}_p)$$
$$(\alpha_0 : \ldots : \alpha_n) \mapsto (\alpha_0^p : \alpha_1^p : \ldots : \alpha_n^p)$$

Moreover: Galois theory $\Longrightarrow$ $V(\mathbb{F}_{p^s})$ = set of fixed pts of φ^s.

Weil: Q What if there's some cohomology theory for $V(\overline{\mathbb{F}}_p)$ s.t. Lefschetz fixed pt formula & other results hold AND corresponds to singular cohomology for $V(\mathbb{C})$?

l-adic Cohomology

Artin-Grothendieck-Verdier came up with such a cohomology theory called l-adic cohomology $H^i(V(\overline{\mathbb{F}}_p), \mathbb{Q}_l) \in \mathrm{Vect}_{\mathbb{Q}_l}$.

N_s = no. fixed pts of φ^s

$$= \sum (-1)^i \operatorname{tr}\Big(\underset{\substack{\parallel \\ \varphi^{*s}}}{\varphi^{s*}} \,\Big|\, \underset{\substack{\parallel \\ H^i}}{H^i(V(\bar{\mathbb{F}}_p), \mathbb{Q}_\ell)}\Big)$$

Lemma: $\exp\left(\sum \operatorname{tr}(\varphi^{*s} | H^i) \frac{t^s}{s}\right)$

$$= \det(\mathrm{id} - t\varphi^* | H^i)^{-1}$$

N.B. The det is a polynomial in t.

Why? We can upper triangularise φ^* & see both sides depend only on e-values $\lambda_1, \ldots, \lambda_n$.

$\operatorname{tr} \varphi^{*s} = \lambda_1^s + \ldots + \lambda_n^s$ so

$$\mathrm{LHS} = \exp\left(\sum_{j=1}^n \sum_s \lambda_j^s \frac{t^s}{s}\right)$$

$$= \prod_{j=1}^n \exp\left(\sum \lambda_j^s \frac{t^s}{s}\right)$$

$$= \prod (1 - \lambda_j t)^{-1} = \mathrm{RHS}$$

$\square$

Cor: If V is d-dim. then

$$Z(V, t) = \frac{P_1(t) P_3(t) \ldots P_{2d-1}(t)}{P_0(t) P_2(t) \ldots P_{2d}(t)}$$

where $P_i(t) = \det(\mathrm{id} - t\varphi^* | H^i(V, \mathbb{Q}_\ell))$

Rem: Weil conjectured $Z(V, t)$ is

rational which cor. ensures. This means fn $s \mapsto N_s$ encoded in finite amount of data.

Q What can you say about $P_i(t)$?

A1 $\deg P_i(t) = \dim_{\mathbb{Q}_\ell} H^i(V(\overline{\mathbb{F}}_p), \mathbb{Q}_\ell)$

$$= \dim_{\mathbb{Q}} H^i(V(\mathbb{C}), \mathbb{Q})$$

$$= \beta_i(V(\mathbb{C})).$$

A2 Weil also conjectured the (analogue of the) Riemann Hypothesis: $P_i(t) \in \mathbb{Z}[t]$ has constant 1 and roots have magnitude $p^{-i/2}$.

Proven by Deligne (70s).