ALGEBRAIC GEOMETRY, D. CHAN

1. Why Algebraic geometry

The basic aim is to solve polynomial equations. Let $\mathbb{K}$ be an algebraically closed field. Geometrical meaning: let $\mathbb{A}^n$ be the set $\mathbb{K}^n$ as an affine space with coordinates $x_1, \ldots, x_n$. Consider $S \subseteq \mathbb{K}[x_1, \ldots, x_n]$ and the zero locus

$$V(S) \quad = \quad \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n \mid \forall f \in S, f(\alpha_1, \ldots, \alpha_n) = 0\}$$

These are called algebraic sets. In algebraic geometry, these and "natural compactifications" are the primary objects of study.
These are interesting in their own right. We outline some examples.

Example 1.1. Let $f(x,y), g(x,y) \in \mathbb{K}[x,y] \rightsquigarrow$plane curves $V(f), V(g) \subseteq \mathbb{A}^2$. When are $V(f)$ and $V(g)$ isomorphic? If $V(f)$ and $V(g)$ are isomorphic, is $\deg(f) = \deg(g)$? If $\mathbb{K} = \mathbb{C}$, then $\dim_{\mathbb{R}}(V(f)) = \dim_{\mathbb{R}}(V(g)) = 2$ in $\mathbb{C}^2$. What is the topology of the curve? What is its natural compactification?

Example 1.2. What is ${}^{\#}(V(f), V(g))$? This is answered by Bezout's theorem. In general ${}^{\#}(V(f), V(g)) = \deg(f) \cdot \deg(g)$.

Example 1.3. Consider 5 plane conics, $c_i = V(f_i(x,y))$ with $\deg(f_i) = 2$ for $i = 1, \ldots, 5$. In general, how many conics are tangent to all 5? We sketch an approach solving this. Consider all quadratics $\{ax^2 + bxy + cy^2 + dx + ey + f \mid a,b,c,d,e,f \in \mathbb{K}\}$. Then the space of conics is five dimensional, since we can divide by any scalar in $\mathbb{K}$. The condition of being tangential to a given conic c means to have a conic on a 4 dimensional hypersurface H_c in the five dimensional space of curves. We want the number of points in $H_{c_1} \cap \ldots \cap H_{c_5}$. The answer is less than 6^5.

Note 1.4. The exact answer is 3264. For calculations see p729 of Griffiths and Harris.

Algebraic geometry also arises naturally in other branches of mathematics.

Example 1.5. Consider solutions of $f(x,y) = x^n + y^n = 1$. We either generalise the setup to $\mathbb{K} = \mathbb{Q}$ and/or $\mathbb{K} = \overline{\mathbb{Q}}$ and consider the action of $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the answer is found by considering the fixed points of the G-action on $\overline{\mathbb{Q}}/\mathbb{Q}$. (Compare with the G-action on $\overline{\mathbb{R}}/\mathbb{R} = \mathbb{C}/\mathbb{R}$).

Example 1.6. Representation theory. When classifying objects in algebra and the answer is not discrete, it is often the case that the (continuous) moduli space is algebraic, or related to an algebraic object.

Example 1.7. Classify the 'two dimensional representations' of $R = \mathbb{K}[x,y]$, i.e. the $\mathbb{K}$-algebra homomorphisms $\varphi : R \longrightarrow \mathbb{K}^{2\times 2}$. It suffices to define the 2×2 matrices $\varphi(x) = (x_{ij})$ and $\varphi(y) = (y_{ij})$, satisfying $[\varphi(x), \varphi(y)] = 0$ since $\mathbb{K}[x,y]$ is a commutative algebra. This imposes 4 polynomial equations on the 8 dimensional parameter space, hence the 'two dimensional representations' are classified by an algebraic set.

Note 1.8. The words 'two dimensional representations' are in quotation marks since we usually seek to classify isometry classes of such maps $\varphi : R \longrightarrow \mathbb{K}^{2\times 2}$. This amounts to factoring out simultaneous conjugation on $\varphi(x)$ and $\varphi(y)$.

Example 1.9. Classical groups such as $GL_n(\mathbb{K})$ and $SL_n(\mathbb{K})$ are algebraic. The Bott Borel Weil theorem realises the representations of compact Lie groups on the cohomology modules of G/B where B is the Borel subgroup of G. The quotient group G/B is often algebraic.

Example 1.10. In complex analysis, consider the 2 valued function $f(x) = \sqrt{x(x-1)(x-2)}$ and the associated Riemann surface S given by $V(y^2 - x(x-1)(x-2))$. Then f is a single valued function on S with value at a point given by y.

2. Duality, Zariski topology and coordinate rings

$\mathbb{K}$ denotes some algebraically closed field, and $\mathbb{A}^n_{x_1,x_2}$ is the affine space. We saw that

$$\mathbb{K}[x_1, \ldots, x_n] \quad \supseteq \quad S \quad \rightsquigarrow \quad V(S) \quad \subseteq \quad \mathbb{A}^n$$

Conversely

$$\mathbb{A}^n \quad \supseteq \quad X \quad \overset{\text{define}}{\rightsquigarrow} \quad \mathcal{I}(X) \quad := \quad \{f \in \mathbb{K}[x_1, \ldots, x_n] \mid f(\alpha) = 0 \text{ if } \alpha \in X\} \quad \subseteq \quad \mathbb{K}[x_1, \ldots, x_n]$$

Example 2.1. Consider $\mathbb{A}^2_{x,y}$ ad $X = \{(0,0)\}$ then $\mathcal{I}(X) = (x,y)$. Note that $\mathcal{I}(X) \lhd \mathbb{K}[x_1, \ldots, x_n]$ since if $f, g \in \mathcal{I}(X)$ and $\alpha \in X$, then $(f+g)(\alpha) = f(\alpha) + g(\alpha) = 0$ implies $f + g \in \mathcal{I}(X)$ etc.

We have a type of algebraic geometry duality

Proposition 2.2. *Let $S, S' \subseteq \mathbb{K}[x_1, \dots, x_n]$ and $X, X' \subseteq \mathbb{A}^n$,*

(1)

$$\{\textit{Subsets of } \mathbb{K}[x_1,\dots,x_n]\} \underset{\mathcal{I}}{\overset{V}{\rightleftarrows}} \{\textit{Subsets of } \mathbb{A}^n\}$$

Then V and $\mathcal{I}$ are reverse inclusions. I.e. $S \subseteq S' \Longrightarrow V(S) \supseteq V(S')$ and $X \subseteq X' \Longrightarrow \mathcal{I}(X) \supseteq \mathcal{I}(X')$

(2) $V(\mathcal{I}(X)) \supseteq X$ and $\mathcal{I}(V(S)) \supseteq S$

(3) $\mathcal{I}(V(\mathcal{I}(X))) = \mathcal{I}(X)$ and $V(\mathcal{I}(V(S))) = V(S)$.

(4) $V(S) = V((S))$, where (S) is the ideal in $\mathbb{K}[x_1, \dots, x_n]$ generated by S.

Proof. (1) Exercise

(2) Pick $x \in X$ and $f \in \mathcal{I}(X)$, so $f(x) = 0 \Longrightarrow x \in V(\mathcal{I}(X))$

(3) Applying $\mathcal{I}$ and 1. to 2. gives $\mathcal{I}(V(\mathcal{I}(X))) \subseteq \mathcal{I}(X)$ but 2. implies the reverse inclusion.

(4) By 1. V we have $V(S) \supseteq V((S))$. For the converse, suppose $x \in V(S)$ and consider $\sum r_i s_i \in (S)$ where $r_i \in \mathbb{K}[x_1, \dots, x_n]$ and $s_i \in S$. Then $s_i(x) = 0$ since $s \in V(S)$ implying $\sum r_i(x)s_i(x) = 0$.

□

Corollary 2.3. *The above restricts to a 1:1 correspondence on* $\operatorname{im}(V) = \{$*algebraic subsets of* $V\}$ $\operatorname{im}(\mathcal{I}) \subset \{$*ideal of* $\mathbb{K}[x_1, \dots, x_n]\}$. *We will see what* $\operatorname{im}(\mathcal{I})$ *is below.*

Example 2.4. $X = (\mathbb{N}, 0) \subseteq \mathbb{A}^2_{x,y}$, then $\mathcal{I}(X) = (y)$ (polynomials only have finite zeroes). $V(\mathcal{I}(X)) \overset{4.}{=} x$-axis. $\mathcal{I}(V(\mathcal{I}(X))) = (y)$.

2.1. **The Zariski topology.**

Lemma 2.5.

(1) Let $I_\alpha \lhd \mathbb{K}[x_1, \dots, x_n]$, then $\bigcap V(I_\alpha) = V(\sum I_\alpha)$

(2) Let $I, J \lhd \mathbb{K}[x_1, \dots, x_n]$, then $V(I) \cup V(J) = V(IJ)$

(3) $V(0) = \mathbb{A}^n$ and $V(\mathbb{K}[x_1, \dots, x_n]) = \emptyset$.

Proof. (1) V reverses inclusions so we have $\supseteq$. The converse is by proposition 2.2-4.

(2) V reverses inclusions implies $\subseteq$. Suppose $x \in V(IJ)$ but $x \notin V(I)$ i.e. $\exists f \in I$ with $f(x) \neq 0$. But $x \in V(IJ)$ implies $\forall g \in J, (fg)(x) = f(x)g(x) = 0$ but $f(x) \neq 0$ so $g(x) = 0$, and $x \in V(J)$.

(3) Trivial.

□

Definition 2.6. *The algebraic subsets of $\mathbb{A}^n$ are the closed sets of a topology on $\mathbb{A}^n$ called the Zariski topology. For any algebraic subset $X \subseteq \mathbb{A}^n$, we have the induced Zariski topology, i.e. $Y \subseteq X$ is closed iff Y is algebraic subset of $\mathbb{A}^n$.*

Example 2.7. The Zariski topology in $\mathbb{A}^1$ is the cofinite topology. Note that it is not Hausdorff.

2.2. **The coordinate ring.**

Restricting polynomial functions to an algebraic subset $X \subseteq \mathbb{A}^n_{x_1,\dots,x_n}$ suggests the following

Definition 2.8. *The coordinate ring of X is*

$$\mathbb{K}[X] \quad := \quad \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}(X)$$

Example 2.9. Consider X =point, and $X = (0, 0) \subseteq \mathbb{A}^2_{x,y}$, then

$$\mathbb{K}[X] \quad = \quad \mathbb{K}[x, y]/\mathcal{I}(X) \quad = \quad \mathbb{K}[x, y]/(x, y) \quad = \quad \mathbb{K}$$

If $\{0\} \subseteq \mathbb{A}^1_x$ we have

$$\mathbb{K}[X] \quad = \quad \mathbb{K}[x]/(x) \quad = \quad \mathbb{K}$$

In $\mathbb{A}^0$ we have $\mathbb{K}[X] \simeq \mathbb{K}$. So $\mathbb{K}[X]$ is 'independent' of the embedding of X and captures the intrinsic geometry.

Example 2.10. For algebraic subsets $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$, then $X \times Y \subseteq \mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$ is an algebraic set, $\mathcal{I}(X \times Y) = (\mathcal{I}(X), \mathcal{I}(Y))$.
$V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2_{x,y}$ and $V(z^2 - 1) \subseteq \mathbb{A}^1_z$, then $V(x^2 + y^2 - 1) \times V(z^2 - 1) = ((x^2 + y^2 - 1), (z^2 - 1))$. Draw picture as exercise, intersect cylinder with two planes, to get 2 circles.

Exercise 1 $\mathbb{K}[X \times Y] \simeq \mathbb{K}[X] \otimes_{\mathbb{K}} \mathbb{K}[Y]$. Warning: in general, not the product topology.

3. Commutative algebra Noetherian rings and Nullstellensatz

Let $\mathbb{K}$ =algebraically closed field R =commutative ring. Proofs will be omitted in this lecture, for more details see course notes on webpage.

Definition 3.1. *A $\mathbb{K}$-algebra R is a ring containing $\mathbb{K}$ such that for $\lambda \in \mathbb{K}$, $r \in R$, $\lambda r = r\lambda$.*

We need a technical

Proposition 3.2. *We say that R is Noetherian if it satisfies one of the following equivalent conditions,*

1. *Every ideal $I \lhd R$ is finitely generated.*
2. *(ACC) Any chain of ideals of R, $I_0 \lhd I_1 \lhd \ldots \lhd R$ stabilises, i.e. for large enough n, $I_n = I_{n+1} = I_{n+2} = \ldots$.*
3. *Every submodule of a finitely generated R module is finitely generated.*

Example 3.3. $R = \mathbb{K}$ is Noetherian.

Theorem 3.4. *R Noetherian $\Longrightarrow R[x]$ is Noetherian.*

Corollary 3.5.

1. *$\mathbb{K}[x_1, \ldots, x_n]$ is Noetherian by induction*
2. *For $X \subseteq \mathbb{A}^n$ an algebraic set $\mathbb{K}[X]$ is Noetherian by 1. and proposition 3.2-2.*

Geometric consequences

1. Given an arbitrary algebraic set $V(I) \subseteq \mathbb{A}^n$, $I \lhd \mathbb{K}[x_1, \ldots, x_n]$ with I finitely generated, say by $\{f_1, \ldots, f_n\}$ implies $V(I) = V(\{f_1, \ldots, f_n\})$. So an algebraic set is the zero locus of finite sets of polynomials.
2. (DCC) For $X \subseteq \mathbb{A}^n$ an algebraic set, any chain of closed subsets stabilises.
3. In fact, X is compact in Zariski topology. Exercise, DCC gives finite intersection property.

3.1. Hilbert's Nullstellensatz.

What is the image of $\mathcal{I}$? From above

$$\mathcal{I}\{\text{subsets of } \mathbb{A}^n\} \quad \longrightarrow \quad \{\text{ideals of } \mathbb{K}[x_1, \ldots, x_n]\}$$

Example 3.6. Let $I = (x^2) \lhd \mathbb{K}[x]$ then $\mathbb{A}^1_x \supsetneq V(I) = X$ and $\mathcal{I}(V(I)) = (x) \supset I$.

Proposition 3.7. *Let $I \lhd R$. The radical of I, denoted $\sqrt{I}$, is*

$$\sqrt{I} \quad := \quad \{f \in R \mid f^n \in I \text{ for } n \gg 0\}$$

$\sqrt{I}$ is an ideal of R containing I. We say that I is radical if $\sqrt{I} = I$.

Proof. Clear $\sqrt{I} \supseteq I$. Suppose $f, g \in \sqrt{I}$ so say $f^m, g^n \in I$, and

$$(f+g)^{m+n} \quad = \quad \sum_{i=0}^{m+n} \binom{m+n}{i} f^i g^{m+n-i} \in I$$

since either f^i and g^{m+n-i} is in I. Similarly if $r \in R$, then $(rf)^m = r^m f^m \in I \Longrightarrow rf \in \sqrt{I}$. □

Theorem 3.8. *(Nullstellenstaz) Let $\mathbb{K}$ be algebraically closed as usual. For $I \lhd \mathbb{K}[x_1, \ldots, x_n]$, $\mathcal{I}(V(I)) = \sqrt{I}$. So $\mathrm{im}(\mathcal{I})$ is the set of radical ideals.*

Proof. (Sketch) $\supseteq$: Suppose $f \in \sqrt{I}$, i.e. $f^r \in I$. Then $\forall x \in V(I)$, $f^r(x) = 0 = (f(x))^r \Longrightarrow f(x) = 0$ and so $f \in \mathcal{I}(V(I))$. □

Geometric consequence

1. V and $\mathcal{I}$ restricts to a 1:1 correspondence

$$\{\text{maximal ideals } I \lhd \mathbb{K}[x_1, \ldots, x_n]\} \quad \underset{\mathcal{I}}{\overset{V}{\rightleftarrows}} \quad \{\text{points in } \mathbb{A}^n\}$$

In particular the maximal ideals are those of the form $\mathcal{I}(\alpha_1, \ldots, \alpha_n) = (x_1 - \alpha_1, x_2 - \alpha_2, \ldots, x_n - \alpha_n)$.

Note that $V(x_1 - \alpha_1, \ldots, x_n - \alpha_n) = (\alpha_1, \ldots, \alpha_n)$. So this point is closed and $\mathcal{I}(\alpha_1, \ldots, \alpha_n) = (x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ since $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ is already a maximal ideal.

Use the correspondence in lecture 2 to get the following

$$\{\text{points of } \mathbb{A}^n\} = \{\text{minimal nonempty closed sets}\} \quad \longleftrightarrow \quad \{\text{maximal radical ideals of } R\}$$

Therefore it suffices to show that every maximal ideal is radical. In fact this is true for any prime ideal $P \lhd R$ for R a commutative ring. Recall that P is prime if R/P is an integral domain, or equivalently $I, J \lhd R$ and $P \supseteq IJ$ implies $I \subseteq P$ or $J \subseteq P$. Let us show that P is radical for P prime. Suppose $f \in \sqrt{P}$, so $f^r \in P$, then $R/P \ni (f+P)^r = 0 + P$, but R/P is an integral domain so $f + P = 0 \Longrightarrow f \in P$ and $P = \sqrt{P}$.

Example 3.9. Correspondence restricts to points of an algebraic subset $X \subseteq \mathbb{A}^n$ and maximal ideals of $\mathbb{K}[X]$,

$$\begin{array}{ccc} \{\text{points of an algebraic set } X\} & \rightleftarrows & \{\text{maximal ideals of } \mathbb{K}[X]\} \\ (\alpha_1, \ldots, \alpha_n) & \longmapsto & \dfrac{(x_1 - \alpha_1, \ldots, x_n - \alpha_n)}{\mathcal{I}(X)} \end{array}$$

4. Primes and Localisation

We explore the geometric meaning of prime ideals.

Definition 4.1. *An algebraic set $X \subseteq \mathbb{A}^n$ is irreducible if it cannot be expressed as a union of two proper closed subsets. Otherwise we say X is reducible.*

Example 4.2. Consider $V(xy) \subseteq \mathbb{A}^2_{x,y}$ then $V(xy) = V(x) \cup V(y)$ so $V(xy)$ is irreducible. Note that $\mathcal{I}(X) = (xy)$ is not prime, since $(x)(y) = (xy) < (x), (y)$.

Proposition 4.3.

$$\{\textit{prime ideals } P \lhd \mathbb{K}[x_1, \ldots, x_n]\} \overset{1:1}{\longleftrightarrow} \{\textit{irreducible closed subsets } X \subseteq \mathbb{A}^n\}$$

Proof. Suppose P is prime and $V(P) = V(I) \cup V(J) = V(IJ)$ (second equality from lecture 2) with $I, J \lhd \mathbb{K}[x_1, \ldots, x_n]$. Then $P = \sqrt{P} = \mathcal{I}(V(P)) = \mathcal{I}(V(IJ)) = \sqrt{IJ} = IJ$ from the Nullstellensatz and that prime ideals are radical. Since P is prime, $P \geqslant I$ or J, and V reverse inclusions so $V(P) \subseteq V(I)$ or $V(J)$, implying V is irreducible. The argument reverses to give the converse. □

Theorem 4.4. *Let X and Y be algebraic sets, then X, Y irreducible implies $X \times Y$ is irreducible.*

Proof. Let $\pi : X \times Y \longrightarrow Y$ be the trivial bundle over Y with typical fibre X, and suppose $X \times Y = Z_1 \cup Z_2$ with Z_i closed algebraic sets. Then $\pi^{-1}(y) = X \times \{y\} := X_y \simeq X$. Decompose using Z_1 and Z_2, we have $\pi^{-1}(y) = (X_y \cap Z_1) \cup (X_y \cap Z_2)$. Since X is irreducible, $X_y \subseteq Z_1$ or Z_2.
Define $Y_i := \{y \in Y \mid X_y \subseteq Z_i\}$, so $Y = Y_1 \cap Y_2$ and if we show Y_1 and Y_2 are closed, then Y irreducible implies $Y \subseteq Y_1$ or Y_2. This gives $X_y \subseteq Z_1$ or Z_2 for all $y \in Y$, so $X \times Y = Z_1$ or Z_2 and we are done.
Finally Y_i is closed since $Y_i = \bigcap_{x \in X} \rho^{-1}(x) \cap Z_i$, where $\rho : X \times Y \longrightarrow X$ a trivial bundle, and intersections of closed sets are closed. □

Theorem 4.5. *Let X be an algebraic set, then X can be written as $X = \bigcup_{i=1}^n X_i$, where X_i is irreducible closed algebraic sets. Moreover if n is minimal, then $X_1, \ldots, X_n$ are uniquely determined and are called the irreducible components of X.*
Alternatively, let $\{X_i\}$ be a family of irreducible closed subsets of X, X_i are irreducible components of X if $X = \bigcup_{i=1}^n X_i$ and $i \neq j \Longrightarrow X_i \nsubseteq X_j$.

Proof. (Sketch) Assume X is not irreducible, then we can write $X = X_1 \cup X_2$ and if X_1 or X_2 is not irreducible, we can continue the process. This procedure terminates due to the DCC. □

Example 4.6. Consider $V(xy, xz, yz) \subseteq \mathbb{A}^3_{x,y,z}$, this is the x, y and z axes. Claim $V(xy, xz, yz) = V(x, y) \cup V(x, z) \cup V(y, z)$, and each of these is irreducible since $\mathbb{K}[x\text{-axis}] = \mathbb{K}[x, y, z]/\mathcal{I}(y, z) \simeq \mathbb{K}[x]$ which is a domain, this implies $\mathcal{I}(y, z)$ is prime and hence the x-axis is irreducible.

4.1. **Localisation.** The aim is to construct a ring which resembles the field of fractions for a domain. Let R be a commutative ring, we first need a set of denominators, this is given by the

Definition 4.7. *A multiplicatively closed subset $T \subseteq R$ is a subset which satisfies*

(1) $s, t \in T \Longrightarrow st \in T$
(2) $1 \in T$

Theorem 4.8. *Let $T \subseteq R$ be a multiplicatively closed subset. Define*

$$R[T^{-1}] = \frac{\{r/t \mid r \in R, t \in T\}}{\sim}$$

where $r/t \sim r'/t' \Longleftrightarrow t''(rt' - r't) = 0$ for some $t'' \in T$. Moreover $R[T^{-1}]$ is a ring with the obvious operations, $\frac{r}{t} + \frac{r'}{t'} = \frac{rt'+r't}{tt'}$ and $\frac{r}{t} \cdot \frac{r'}{t'} = \frac{rr'}{tt'}$. Further, we have a ring homomorphism $R \longrightarrow R[T^{-1}]$ defined by $r \longmapsto \frac{r}{1}$.

Proof. Routine checking. □

Note 4.9. If $0 \in T$, then $R[T^{-1}] = 0$ and if R is an integral domain the condition $t''(rt' - r't) = 0$ is the same as $rt' - r't = 0$.

Example 4.10. Let $T = \{1, t, t^2, \ldots\}$ for some $t \in R$. Then T is multiplicatively closed. We write $R[f^{-1}] := R[T^{-1}]$ and $R[f^{-1}] = \frac{R[y]}{(fy-1)}$ with $y = f^{-1}$.

Example 4.11. Let $P \lhd R$ be a prime ideal, then clearly $T := R - P$ is multiplicatively closed and we denote $R_P := R[T^{-1}]$.

5. Regular functions and maps

Definition 5.1. *An irreducible algebraic set $X \subseteq \mathbb{A}^n$ is called an affine variety.*

In the sequel, let $\mathbb{K}$ be algebraically closed and X an affine variety.
In this case, $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}(X)$ where $\mathcal{I}(X)$ is prime, so $\mathbb{K}[X]$ is a domain. Let $T = \{f \in \mathbb{K}[X] \mid f \neq 0\}$, then T is a multiplicatively closed set. Define $\mathbb{K}(X) := \mathbb{K}[X][T^{-1}]$, called the function field, or the ring of rational functions. Moreover, it is a field.

Proposition 5.2. *A rational function $f \in \mathbb{K}(X)$ is regular at $x \in X$ if the following equivalent conditions hold,*

(1) $f = g/h$ for some $g, h \in \mathbb{K}[X]$, with $h(x) \neq 0$.
(2) $f \in \mathbb{K}[X]_{\mathcal{M}_x} \subseteq \mathbb{K}(X)$, where $\mathcal{M}_x$ is the maximal ideal corresponding to x.

Proof. Let $x = (\alpha_1, \ldots, \alpha_n)$, then $\mathcal{M}_x = (x_1 - \alpha_1, \ldots, x_n - \alpha_n) = \ker(\varphi_x)$ where $\varphi_x : \mathbb{K}[X] \longrightarrow \mathbb{K}$ is defined by $f \longmapsto f(x)$. So $h(x) \neq 0 \iff h \notin \mathcal{M}_x$. □

Let $U \subseteq X$ be an open set of an affine variety, then we say U is a quasi-affine variety. Note that U is either dense or empty, since X is irreducible and $X = \overline{U} \cup (X - U)$.

Definition 5.3. *A regular function on U is a regular function $f \in \mathbb{K}[X]$ which is regular for every $x \in U$.*

Example 5.4. Let $X = \mathbb{A}^2_{x,y}$ and $U = X - V(y)$, then $\frac{x^3+2xy+y^2}{y^3}$ is regular.

Definition 5.5. *A hypersurface in $\mathbb{A}^n$ is an algebraic set of the form $V(f)$ for some $f \in \mathbb{K}[x_1, \ldots, x_n]$. If X is an affine variety, a principal open set is of the form $X - V(f)$, denoted $D(f)$. Note that f is nonzero in $D(f)$.*

Example 5.6. These principal open sets form a basis for the Zariski topology.

Theorem 5.7. *Consider principal open sets $D(f) \subseteq X$ an affine variety. The set of regular functions on $D(f)$ is $\mathbb{K}[X][f^{-1}] \subseteq \mathbb{K}(X)$. In particular, the regular functions on X is the set $\mathbb{K}[X]$.*

Proof. Suppose $g : D(f) \longrightarrow \mathbb{K}$ is regular, i.e. $\forall x \in D(f)$, $g = g'_x/g''_x$ where $g''_x(x) \neq 0$.
Recall that X Noetherian $\overset{\text{exercise}}{\longrightarrow}$ $D(f)$ Noetherian$\Longrightarrow D(f)$ compact. We have $D(f) \subseteq \bigcup_{x \in D(f)} D(g''_x)$ and since $D(f)$ is compact, $D(f) \subseteq \bigcup_{i=1}^n D(g''_{x_i})$ for suitable $x_1, \ldots, x_n \in D(f)$.
Taking complements and intersecting with X, we have

$$V(\mathcal{I}(X) + (f)) \quad = \quad X \cap V(f) \quad \supseteq \quad X \cap \left(\bigcap_{i=1}^n V(g''_{x_i}) \right) \quad = \quad V(\mathcal{I}(X) + (g''_{x_1}, \ldots, g''_{x_n}))$$

Applying $\mathcal{I}$ on both sides and the Nullstellensatz give,

$$\mathcal{I}(X) + (f) \quad \subseteq \quad \sqrt{\mathcal{I}(X) + (f)} \quad \subseteq \quad \sqrt{\mathcal{I}(X) + (g''_{x_1}, \ldots, g''_{x_n})}$$

so $f \in \sqrt{\mathcal{I}(X) + (g''_{x_1}, \ldots, g''_{x_n})}$ and by definition of radicals, $\exists s \gg 0$ such that $f^s = \sum_{i=1}^n a_i g''_{x_i}$, with $a_i \in \mathbb{K}[X]$. Now

$$f^s \quad = \quad \sum_{i=1}^n a_i \frac{g'_{x_i}}{g} \quad \Longleftrightarrow \quad g \quad = \quad \sum_{i=1}^n a_i \frac{g'_{x_i}}{f^s} \quad \in \quad \mathbb{K}[X][f^{-1}]$$

The last assertion follows by substituting $f = 1$. □

5.1. Regular maps.

Let U be a quasi-affine variety, then a regular map, or a morphism of varieties is a map $\varphi : U \longrightarrow \mathbb{A}^m$ of the form $(\varphi_1, \ldots, \varphi_m)$ where $\varphi_1, \ldots, \varphi_m$ are regular functions on U. If Y is another quasi-affine variety containing $\mathrm{im}(\varphi)$, then we say that φ is a regular map, or morphism of varieties $\varphi : U \longrightarrow Y$.

Example 5.8. $\varphi : \mathbb{A}^1_t = U \longrightarrow Y \subseteq \mathbb{A}^2_{x,y}$, defined by $t \longmapsto (t^2, t^3)$. In fact, we can take $Y = (y^2 - x^3)$ and $\varphi : U \longrightarrow Y$ is a homeomorphism. But U and Y are not isomorphic as varieties, (Y has a singular point, U does not).

Example 5.9. $\pi : \mathbb{A}^2_{x,y} \longrightarrow \mathbb{A}^1_z$ be the usual projection onto the x coordinate.

Example 5.10. Compositions of regular maps are regular (exercise). E.g. $\mathbb{A}^1_t \overset{\varphi}{\longmapsto} \mathbb{A}^2_{x,y} \overset{\psi}{\longmapsto} \mathbb{A}^1_z$ where φ is from the above example, and $\psi : (x, y) \longmapsto f(x, y)$ for some $f \in \mathbb{K}[x, y]$. Then $\psi \circ \varphi$ is a polynomial $\mathbb{K}[t]$.

Example 5.11. The identity map is regular.

Definition 5.12. *A morphism of varieties $\varphi : U \longrightarrow Y$ is an isomorphism of varieties if there is an inverse morphism, i.e. $\exists \psi \in \mathrm{Hom}_{\mathit{Var}}(Y, U)$ satisfying $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_U$.*

Example 5.13. Let $\varphi : \mathbb{A}^1_t - \{0\} := U \longrightarrow \mathrm{im}(\varphi) =: Y \subseteq \mathbb{A}^2_{x,y}$ defined by $t \longmapsto (t, 1/t)$. This is an isomorphism with inverse $\psi : Y \longrightarrow \mathbb{A}^2_{x,y} \overset{\pi}{\longmapsto} \mathbb{A}^1$ defined by $(x, y) \longmapsto x$. Check that φ and ψ are morphisms.$\mathbb{K}[U] = \mathbb{K}[t, t^{-1}]$, $\mathbb{K}[x, y]/(xy - 1) \simeq \mathbb{K}[x, x^{-1}]$.

6. Duality and continuity

Remark 6.1. The Nullstellensatz implies that the coordinate rigs of affine varieties are the finitely generated $\mathbb{K}$-algebras which are integral domains. We extend this to the corresponding maps.

6.1. Functoriality.

Proposition 6.2. *Consider a regular map of affine varieties, $\varphi : X \longrightarrow Y$, then the pullback $\varphi^* : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$ is a well defined algebra homomorphism.*

Proof. Check φ^* is a homomorphism

$$(f + g)^*(x) = (f + g)(\varphi(x)) = (f \circ \varphi)(x) + (g \circ \varphi)(x) = (f^* + g^*)(x)$$

similarly for $(f \cdot g)^* = f^* \cdot g^*$. Check $f \circ \varphi \in \mathbb{K}[X]$ which is clear for any example,

$$\begin{aligned} \varphi : \mathbb{A}^1_t &\longrightarrow \mathbb{K}[t] \\ f(x, y) &\longmapsto f(t^2, t^3) \quad \in \mathbb{K}[t] \end{aligned}$$

explicitly, $x \longmapsto t^2, y \longmapsto t^3$. □

Proposition 6.3. *Consider regular maps of affine varieties $X \overset{\varphi}{\longrightarrow} Y \overset{\psi}{\longrightarrow} Z$, then there exists an inverse ψ*

(1) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$
(2) $\mathrm{id}^* = \mathrm{id}$

Example 6.4. Let φ be as in proof of proposition 6.2. We show φ is not an isomorphism. Suppose it is, and let ρ be its inverse.

$$\rho^* \circ \varphi^* = (\varphi \circ \rho)^* = \mathrm{id}$$

since $\varphi^* \circ \rho^*$ is an isomorphism, so is φ^*. But this implies $\mathrm{im}(\varphi^*) = \mathbb{K}[t^2, t^3] \subset \mathbb{K}[t]$, in particular $im(\varphi^*) \neq \mathbb{K}[t]$, contradiction. This example shows that bijective regular maps need not be isomorphisms.

Proposition 6.5.

(1) The following is a bijection for X, Y affine varieties.

$$\begin{aligned} \{\textit{regular maps } X \longrightarrow Y\} &\longleftrightarrow \{\mathbb{K}\textit{-algebra homomorphism } \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]\} \\ \varphi &\longmapsto \varphi^* \end{aligned}$$

(2) The category of affine varieties is equivalent to the opposite category of finitely generated $\mathbb{K}$-algebras which are domains.

Proof. We construct an inverse to $\varphi \longmapsto \varphi^*$. Let $X \subseteq \mathbb{A}^n_{x_1,\dots,x_n}, Y \subseteq \mathbb{A}^m_{y_1,\dots,y_m}$, consider

$$\mathbb{K}[y_1, \dots, y_m]/\mathcal{I}(Y) = \mathbb{K}[Y] \longrightarrow \mathbb{K}[X] = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}(X)$$

Define $\varphi : X \longrightarrow Y$ by $(x_1, \dots, x_n) \longmapsto (\psi(y_1)(x_1, \dots, x_n), \dots, \psi(y_m)(x_1, \dots, x_n))$ for any $\psi : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$, where $y_1, \dots, y_m \in \mathbb{K}[Y]$. We check that $im(\varphi) \subseteq Y$: $\forall f \in \mathcal{I}(Y)$, $f(\psi(y_1), \dots, \psi(y_n)) = \psi(f(y_1, \dots, y_n)) = 0$ since ψ is a homomorphism. Exercise: show that this construction is inverse to $\varphi \longmapsto \varphi^*$. □

Remark 6.6.

$$\{\text{algebraic subsets } Y \subset \mathbb{A}^n_{x_1,\dots,x_n}\} \longleftrightarrow \{\text{radical ideals } I \lhd \mathbb{K}[x_1, \dots, x_n]\}$$

restricts to

$$\{\text{closed subsets of } X\} \longleftrightarrow \{\text{radical ideals of } \mathbb{K}[X]\}$$

6.2. **Continuity.**

Proposition 6.7. *Let $\varphi : U \longrightarrow Y$ be a regular map of quasi-affine varieties then φ is continuous.*

We will need the following lemma.

Lemma 6.8. *A principal open subset, $D(f)$ of an affine variety X is itself an affine variety.*

Proof. Exercise: sketch Z irreducible $\Longleftrightarrow$ (V dense open subset of topological space X then V irreducible $\Longleftrightarrow$ Z irreducible). □

Proof. (of proposition 6.7) We saw that the ring of regular functions on $D(f)$ is $\mathbb{K}[X][f^{-1}] \subset \mathbb{K}(X)$, which is finitely generated ($\mathbb{K}[X]$ is finitely generated and we add 1 generator) and a domain (subring of a field $\mathbb{K}(X)$). By above $D(f)$ is an affine variety.
Since the sets $D(f)$ is a basis for the topology, it suffices to assume $X = U$ affine and $Y = \mathbb{A}^m$. The answer follows from the claim

$$\varphi^{-1}(V(I)) \quad = \quad V(\varphi^*(I))$$

this is true because, $x \in \varphi^{-1}(V(I)) \Longleftrightarrow \varphi(x) \in V(I) \Longleftrightarrow \forall f \in I, f(\varphi(x)) = 0 = (\varphi^* f)(x) \Longleftrightarrow x \in V(\varphi^*(I))$. □

Example 6.9. Consider $\mathbb{A}^3_{x_1,x_2,x_3} \longrightarrow^{\pi} \mathbb{A}^2_{x_1,x_2}$, then $\varphi^* f = f(x_1, x_2)$.

7. Quasi-projective Varieties I

Definition 7.1. *Let W be a finite dimensional vector space over $\mathbb{K}$. Then $\mathbb{K}^*$ acts on W via scalar multiplication. Define $\mathbb{P}(W) := (W - \{0\})/\mathbb{K}^*$ = sets of lines in W containing 0. If $W = \mathbb{K}^{n+1}$, then $\mathbb{P}^n := \mathbb{P}(W)$. Write $(x_0 : \ldots : x_n)$ for the $\mathbb{K}^*$-orbit of $(x_0, \ldots, x_n)$. These are called homogeneous coordinates.*

Proposition 7.2. *A polynomial $p \in \mathbb{K}[x_1, \ldots, x_n]$ is homogeneous of degree d if all monomials are of degree d. Denote the set of all degree d homogeneous polynomials, $\mathbb{K}[x_1, \ldots, x_n]_d$. In this case, Euler's relation holds: for $\lambda \in \mathbb{K}$,*

$$f(\lambda x_1, \ldots, \lambda x_n) \quad = \quad \lambda^d f(x_1, \ldots, x_n)$$

Hence, the set where $f = 0$ is independent of the choice of homogeneous coordinates. So for any $S \subseteq \mathbb{K}[x_1, \ldots, x_n]_d$, it makes sense to define $V(S) \subseteq \mathbb{P}^n$ to be $\{(x_0 : \ldots : x_n) \mid f(x_0, \ldots, x_n) = 0\}$.

7.1. **Homogeneous ideals.** Let $R := \mathbb{K}[x_1, \ldots, x_n] = \bigoplus_{d=0}^{\infty} R_d$ where $R_d = \mathbb{K}[x_1, \ldots, x_n]_d \cup \{0\}$. Note that R_d is a finite dimensional vector space.

Proposition 7.3. *$I \lhd R$ is a homogeneous ideal if either of the following equivalent properties hold,*

1. *$I \overset{v.s.}{=} \bigoplus I_d$ where $I_d \subseteq R_d$.*
2. *$\forall f \in I$, f has a decomposition w.r.t. to the grading $f = f_d + f_{d+1} + \ldots + f_e$ with each $f_i \in I$.*

Proof. $1 \Longrightarrow 2$ is clear. For $2 \Longrightarrow 1$, $I = \bigoplus R_d \cap I$. □

Example 7.4. $V(z^2 - x^2 - y^2) \subseteq \mathbb{P}^2_{x,y,z}$, the ideal $z^2 - x^2 - y^2$ is homogeneous. The ideal $(y - x^2)$ is not homogeneous. Since $y - x^2$ decomposes into $f_1 = y$ and $f_2 = x^2$, but neither of these are in $(y - x^2)$. We can think of homogeneous ideals as the ones generated by homogeneous polynomials.

7.2. **Topology of $\mathbb{P}^n$.** This is completely analogous to the affine case.

Proposition 7.5. *For a homogeneous ideal $I = \bigoplus I_d \lhd \mathbb{K}[x_1, \ldots, x_n]$. Define $V(I) = V(\bigcup I_d) \overset{?}{=} \bigcup V(I_d)$. Then*

1. *For $S \subseteq \mathbb{K}[x_1, \ldots, x_n]_d$, the ideal generated by S is homogeneous and $V((S)) = V(S)$.*
2. *The $V(I)$ for homogeneous ideals $I = (S)$ form the closed sets of the topology.*
3. *Any closed set is in the form $V(S)$ for some finite set $S \subseteq \mathbb{K}[x_1, \ldots, x_n]_d$.*
4. *$\mathbb{P}^n$ is Noetherian and compact in the Zariski topology.*

Remark 7.6. Note that the concepts of Noetherian topological spaces, irreducibility, irreducible components, DCC on closed sets, etc depend only on the topology so makes sense for $\mathbb{P}^n$ by part 2.

Definition 7.7. *A closed irreducible subset $X \subseteq \mathbb{P}^n$ is called a projective variety. An open subset of X is called a quasi-projective variety.*

7.3. **Affine pieces.** Consider $\mathbb{P}^n_{x_1,\dots,x_n}$, define an open subset $U_i := \mathbb{P}^n - V(x_i)$ called an affine piece of $\mathbb{P}^n$. This allows working affine locally.

Example 7.8. Consider $f(x,y,z) = y^2z - x^3 + xz^2 \in \mathbb{K}[x,y,z]_3$ and the closed subset of $\mathbb{P}^2_{x,y,z}$. Look at the affine piece $U_z := \{(X : Y : 1) \mid X, Y \in \mathbb{K}\}$ where $X = x/z, Y = y/z$. X and Y are like affine coordinates. We have the local bijection $U_2 \xrightarrow{\sim} \mathbb{A}^2_{x,y}$ defined by $(X : Y : 1) \longmapsto (X, Y)$. To get $V(f)$ affine locally, we dehomogenise f by $F(X,Y) := \frac{1}{z^{\deg(f)}} f(x,y,z) = Y^2 - X^3 + X$ (essentially set $z = 1$). Therefore $V(f) \cap U_z$ corresponds to points of $\mathbb{A}^2_{x,y}$ such that the dehomogenised polynomial $F(X,Y) = 0$. The reverse procedure can also be performed. Given $G(X,Y)$, we can homogenise by setting $g(x,y,z) = z^{\deg(g)} G\left(\frac{x}{z}, \frac{y}{z}\right)$.
The converse result: let $U := \{(X : Y : 1) \mid g(X,Y) = 0\} \subseteq U_z \subset \mathbb{P}^2$. What is $\overline{U} \subseteq \mathbb{P}^2$? Exercise: $\overline{U} = V(S)$ where S is a set of homogeneous polynomials which are homogenisations of polynomials in $(g(X,Y))$.

Proposition 7.9. *$\mathbb{P}^n$ is covered by open affine patches $U_i := \mathbb{P}^n_{x_0,\dots,x_n}$. The map $\Phi : U_i \longrightarrow \mathbb{A}^n$ given by $(x_0 : \dots : x_n) \longmapsto \left(\frac{x_1}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right)$ is a homeomorphism.*

In fact, $V := V(I) \longrightarrow \overline{\Phi^{-1}(V)}$ sets up a correspondence

$$\begin{array}{ccc} \{\text{closed subsets of } \mathbb{A}^n \simeq U_j\} & \longleftrightarrow & \{\text{closed sets of } \mathbb{P}^n\} \\ \Phi(U \cap U_j) & \longleftarrow & U = V(S) \end{array}$$

where $S \subseteq \mathbb{K}[x_1,\dots,x_n]_d$. This is given by homogenising polynomials in I and dehomogenising polynomials in S. Hence a projective variety $X \subseteq \mathbb{P}^n$ is covered by affine varieties, $\{X \cap U_i\}$, called the affine patches of X.

8. Grassmannians

Definition 8.1. *The Grassmannian of r-dimensional subspaces of $\mathbb{K}^n$ is the set*

$$G(r,n) \quad := \quad \{V \leqslant \mathbb{K}^n \mid \dim(V) = r\}$$

For $V \in G(r,n)$, $\mathbb{P}(V) = \frac{V - \{0\}}{\mathbb{K}^*} \subseteq \mathbb{P}^{n-1}$ is called an $(r-1)$-plane in $\mathbb{P}^{n-1}$. Hence $G(r,n)$ also classifies the $(r-1)$ planes in $\mathbb{P}^{n-1}$.

Note 8.2. Consider $V \leqslant \mathbb{K}^n$ as above. Then $\bigwedge^r V$ is a line containing 0 in $\bigwedge^r \mathbb{K}^n$, therefore defines a point in $\mathbb{P}(\bigwedge^r \mathbb{K}^n)$. Hence we have a map $G(r,n) \longrightarrow \mathbb{P}(\bigwedge^r \mathbb{K}^n)$. This makes $G(r,n)$ a closed subset of $\mathbb{P}(\bigwedge^r \mathbb{K}^n)$ if $x \in \bigwedge^r \mathbb{K}^n$ satisfies the Plücker relations,

$$i(u)x \wedge x \quad = \quad 0 \quad \text{for all } u \in \bigwedge{}^{r-1} \mathbb{K}^{n*}$$

where $i(u) : \bigwedge^r \mathbb{K}^n \longrightarrow \mathbb{K}^{n*}$ is the contraction operator defined by $(i(u)x)(v) = x(u \wedge v)$ for all $v \in \mathbb{K}^{n*}$.
For $r = 2$, this formula specialises to the following. Write $x = \sum_{k,l} p_{kl} e_k^* \wedge e_l^* \in \bigwedge^2 \mathbb{K}^n$, it suffices to check for the basis $\{e_1^*, \dots, e_n^*\} \subseteq \mathbb{K}^{n*}$. Recall $(i(e_j^*)x)(e_i^*) = x(e_j^* \wedge e_i^*) = p_{ji}$,

$$0 \quad = \quad i(e_j^*)x \wedge x \quad = \quad \sum_i p_{ji} e_i^* \wedge \sum_{k,l} (p_{kl} e_k^* \wedge e_l^*) \quad = \quad \sum_{i,j,k} p_{ji} p_{kl} e_i^* \wedge e_k^* \wedge e_l^*$$

Now for every triple (s,t,u), $s,t,u = 1,2,3$; s,t,u distinct; and $1 \leqslant l_s < l_t < l_u \leqslant n$. We get,

$$\sum_{s=1, j<l_s}^{3} p_{jl_s} p_{l_t l_u} (-1)^{s+1} - \sum_{s=1, j>l_s}^{3} p_{l_s j} p_{l_t l_u} (-1)^{s+1} \quad = \quad 0 \tag{1}$$

This formula has to hold for all $j \in \{1, \dots, n\}$ and triples $1 \leqslant l_1 < l_2 < l_3 \leqslant n$.

8.1. **Elementary treatment of above.** Let $V \in G(r,n)$ and $\{b_1, \dots, b_r\}$ be a basis for V. Let

$$A \quad = \quad \begin{pmatrix} \boldsymbol{b}_1^T \\ \vdots \\ \boldsymbol{b}_r^T \end{pmatrix} \quad = \quad \begin{pmatrix} \boldsymbol{a}_1 & \dots & \boldsymbol{a}_n \end{pmatrix}$$

For each r-tuple $(i_1, \dots, i_r)$ with $1 \leqslant i_1 < \dots < i_r \leqslant n$ define Plücker coordinates of A to be

$$p_{i_1,\dots,i_r}(A) \quad = \quad \det \begin{pmatrix} \boldsymbol{a}_{i_1} & \dots & \boldsymbol{a}_{i_r} \end{pmatrix}$$

Then if $N := \binom{n}{r} - 1 =$ number of such r-tuples -1, we let $(p_{i_1,\dots,i_r})$ be coordinates in $\mathbb{P}^N$ and $(p_{i_1,\dots,i_r}(A))$ is a point in $\mathbb{P}^N$. If we change basis for V, i.e. we multiply A on the left by some $M \in GL_r$, then $p_{i_1,\dots,i_r}(A)$ changes by a factor of $\det(M)$ for all r-tuples $(i_1, \dots, i_r)$, with $1 \leqslant i_1 < \dots < i_r \leqslant n$. This gives a well defined map $\pi : G(r,n) \longrightarrow \mathbb{P}^N$.
We will prove the following theorem for $r = 2$.

Theorem 8.3. *π is injective and $\operatorname{im}(\pi)$ is closed in $\mathbb{P}^N$. Moreover the affine pieces are $\mathbb{A}^{r(n-r)}$.*

Lemma 8.4. $\mathrm{im}(\pi)$ *satisfies the specialised Plücker relations (1).*

Proof. Consider A as before, assume $j < l_1 < l_2 < l_3$, let

$$\boldsymbol{b} = \begin{pmatrix} -a_{2j} \\ a_{1j} \end{pmatrix}$$

then the Plücker coordinates of A is $p_{j,l_s} = \begin{vmatrix} \boldsymbol{a}_j & \boldsymbol{a}_{l_s} \end{vmatrix} = \boldsymbol{b} \cdot \boldsymbol{a}_{l_s}$, then the *r.h.s.* of (1) is

$$\begin{vmatrix} \boldsymbol{b}\cdot\boldsymbol{a}_{l_1} & \boldsymbol{b}\cdot\boldsymbol{a}_{l_2} & \boldsymbol{b}\cdot\boldsymbol{a}_{l_3} \\ \boldsymbol{a}_{l_1} & \boldsymbol{a}_{l_2} & \boldsymbol{a}_{l_3} \end{vmatrix} \overset{\text{row operations}}{=} \begin{vmatrix} 0 & 0 & 0 \\ \boldsymbol{a}_{l_1} & \boldsymbol{a}_{l_2} & \boldsymbol{a}_{l_3} \end{vmatrix} = 0$$

□

The theorem will follow from

Lemma 8.5. *($r = 2$) Given Plücker coordinates (p_{st}) with affine piece $p_{12} = 1$ (the other cases are identical) satisfying (1), then there exists a unique $V \in G(r,n)$ such that V has Plücker coordinates (p_{st}). In fact V is given by*

$$A = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \boldsymbol{a}_3 & \dots & \boldsymbol{a}_n \end{pmatrix}$$

where $a_{1s} = -p_{2s}$ and $a_{2s} = p_{1s}$, $s \geqslant 3$. The nonconstant entries of A give the $2(n-2)$ affine coordinates.

Proof. Note that $(p_{st}(A)) = (p_{st})$ iff $s = 1$ or 2. Also the choice of $\boldsymbol{a}_s$ is uniquely determined if we assume a is normalised, i.e. $\begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{pmatrix} = I$. Consider (1) when $j = 1, l_1 = 2, l_2 = s, l_3 = t$, we get

$$0 = p_{12}p_{st} - p_{1s}p_{2t} + p_{1t}p_{2s}$$

so we can solve uniquely for p_{st}. □

9. Quasi-projective varieties II

9.1. **Homogeneous coordinate rings.** Let Y be a closed subset of $\mathbb{P}^n_{x_0,\dots,x_n}$, we define an homogeneous ideal as an ideal, $\mathcal{I}(Y) = \bigoplus \mathcal{I}(Y)_d$ ($\mathcal{I}(Y)_d \in \mathbb{K}[x_0,\dots,x_n]_d$) generated by homogeneous polynomials $f \in \mathbb{K}[x_0,\dots,x_n]$ with $f(y) = 0$ for all $y \in Y$. The homogeneous coordinate ring of U is $S(Y) := \mathbb{K}[x_0,\dots,x_n]/\mathcal{I}(Y) := \bigoplus S(Y)_d$, where $S(Y)_d := \mathbb{K}[x_0,\dots,x_n]_d/\mathcal{I}(Y)_d$.

9.2. **Rational and regular functions.** Let Y be a quasi-affine variety. As in the affine case, $\overline{Y}$ is irreducible implies $T :=$ set of nonzero homogeneous elements in $S(\overline{Y})$ is a multiplicatively closed set, consisting of nonzero divisors. We define the ring of rational functions of Y, or the function field of Y to be

$$\mathbb{K}(Y) := \left\{\frac{g}{h} \mid g,h \in S(\overline{Y})_d, h \neq 0\right\} \subseteq S(\overline{Y})[T^{-1}]$$

This is a field which is a subring of $S(\overline{Y})[T^{-1}]$ (exercise).

Note 9.1. The values of g and h are defined up to scalar multiplication by λ^d. Therefore g/h is a well defined function on $Y - \{\text{zeroes of } h\}$.

Definition 9.2. *$f \in \mathbb{K}(Y)$ is regular at $y \in Y$ if we can write $f = g/h$ where $h(y) \neq 0$.*

9.3. **Compatibility of affine pieces.** Let $Y \subseteq \mathbb{P}^n_{x_0,\dots,x_n}$ be a projective variety, and $U_i = Y - V(x_i)$ be the i-th affine piece. Then we have the following isomorphism

$$\begin{aligned} \mathbb{K}(Y) &\xrightarrow{\sim} \mathbb{K}(U_i) \\ \frac{g}{h} &\longmapsto \frac{g/x_i^d}{h/x_i^d} \end{aligned}$$

where $g,h \in S(Y)_d$.

Proof. Recall that U_i is defined by an ideal $\mathbb{K}\left[\frac{x_0}{x_i},\dots,\widehat{\frac{x_i}{x_i}},\dots,\frac{x_n}{x_i}\right]$ obtained by dehomogenising $\mathcal{I}(Y)$. Hence $\mathbb{K}[U_i]$ is a subring of $\mathbb{K}(Y)$ given by all polynomial functions $\left\{\frac{g}{x_i^d} \mid g \in S(Y)_d\right\}$. Localising gives the isomorphism. □

Remark 9.3. Under this identification the concept of regularity at $y \in U_i$ is the same in both the affine and projective cases.

Definition 9.4. *(This definition is nonstandard) Let X be a quasi-projective variety. An open subset $U \subseteq X$ is called an affine open set if it is a principal open subset of some affine piece of $\overline{U}$.*

9.4. **Regular maps.** Let X and Y be quasi-projective varieties. A continuous function $\varphi : X \longrightarrow Y$ is regular if any of the following equivalent conditions hold.

(1) If for any affine open $V \subseteq Y$ and affine open $U \subseteq \varphi^{-1}(V)$, the map of affine varieties $U \longrightarrow V$ is regular.
(2) For any cover $\{U_i\}$ of Y by affine opens, and the cover $\{U_{ij}\}$ of $\varphi^{-1}(V_i)$ by affine opens, every $U_{ij} \longrightarrow V_i$ is regular.
(3) For any $V \subseteq Y$ open and regular function f on V, the pullback function $\varphi^* f$ on $\varphi^{-1}(V)$ is also regular.

Note 9.5. The key point is that regular maps of affine varieties is defined by regular functions and this is independent of the choice of the affine open cover.

Definition 9.6. *Let X and Y be quasi-projective varieties. A rational map $\varphi : X \rightsquigarrow Y$ is a regular map $U \longrightarrow Y$ where U is a dense open (i.e. nonempty) subset of X. We usually identify rational maps if they agree on a dense open subset.*

Example 9.7. Rational maps to $\mathbb{P}^n$ via forms. Let X be a projective variety in $\mathbb{P}^n_{x_0,\dots,x_n}$ and $f_0, \dots, f_n \in S(X)_d$. Then the map $\varphi : X \longrightarrow \mathbb{P}^n$ defined by $x \longmapsto (f_0(x) : \dots : f_n(x))$ is a well defined regular map on $X - V(f_1, \dots, f_n)$. Over the affine piece $Y_i \neq 0$, it is a rational map, $\left(\frac{f_1}{f_i} : \dots : \widehat{\frac{f_i}{f_i}} : \dots : \frac{f_n}{f_i}\right)$.

Example 9.8. Let $X = V(f) \subseteq \mathbb{P}^2_{x,y,z}$ where $f(x,y,z) = x^2 + y^2 + z^2$. The map $X \longrightarrow \mathbb{P}^1_{u,v}$ defined by $(x : y : z) \longmapsto (u : v) = (x : y)$ is a regular map since if $x = y = 0$ then $f = 0 \Longrightarrow z = 0$. This is the projection from the point $(0 : 0 : 1)$. On affine pieces $\mathbb{A}^2_{\frac{y}{x},\frac{z}{x}} \supseteq U_x = \varphi^{-1}(V_u) \longrightarrow V_u \simeq \mathbb{A}^1_{\frac{v}{u}}$, the coordinate ring is $\mathbb{K}\left[\frac{y}{x}, \frac{z}{x}\right] / \left(1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2\right)$, and the map is given by $\left(\frac{y}{x}, \frac{z}{x}\right) \longmapsto \frac{v}{u} = \frac{y}{x}$, the projection onto the first factor on the affine piece. Also $\mathbb{A}^2_{\frac{x}{z},\frac{y}{z}} \supseteq U_z \supseteq \varphi^{-1}(V_u) \cap U_z \longrightarrow V_u \simeq \mathbb{A}^1_{\frac{v}{u}}$, defined by $\left(\frac{x}{z}, \frac{y}{z}\right) \longmapsto \frac{v}{u} = \frac{y}{x} = \frac{y/z}{x/z}$.

10. Birational maps and products

Let X and Y be quasi-projective varieties.

Lemma 10.1. *Suppose $f, g : X \longrightarrow Y$ are regular maps which agree on some open dense subset $U \subseteq X$. Then $f = g$.*

Proof. We require $f(x) = g(x)$ for all $x \in X$. We can assume $Y = \mathbb{P}^n$ and pick a hyperplane avoiding $f(X)$ and $g(X)$ and shrinking X?, if necessary, we can assume $Y = \mathbb{A}^n$. Let $\varphi = (f, g) : X \longrightarrow \mathbb{A}^n \times \mathbb{A}^n$ given by $u \longmapsto (f(u), g(u))$ and $\Delta := \{(u,v) \mid u = v\} \subseteq \mathbb{A}^n \times \mathbb{A}^n$ is a closed subset. Now $\varphi^{-1}(\Delta) \supseteq U$ is closed implies $\varphi^{-1}(\Delta) = X$ i.e. $f = g$. □

Definition 10.2. *A rational map $\varphi : X \rightsquigarrow Y$ is dominant if its image is dense in Y.*

In this case we can find affine open subsets, $\emptyset \neq U \subseteq X, \emptyset \neq V \subseteq Y$ such that φ restricts to a regular dominant map.

Lemma 10.3. *In this case, $\varphi^* : \mathbb{K}(V) \longrightarrow \mathbb{K}(U)$ is injective. Hence this is an induced field homomorphism $\varphi^* : \mathbb{K}(V) \longrightarrow \mathbb{K}(U)$ given by $g/h \longmapsto \varphi^* g / \varphi^* h$.*

Proof. Let $h \in \mathbb{K}[V]$ and V' the dense open set where $h \neq 0$. $\mathrm{im}(\varphi)$ is dense implies $\varphi^{-1}(V') \neq \emptyset$. But $\varphi^* h$ is nonzero on $\varphi^{-1}(V')$, therefore $\varphi^* h \neq 0$. □

10.1. Birational maps.

Definition 10.4. *A rational map $\varphi : X \rightsquigarrow Y$ is birational if it has a rational inverse $\psi : Y \rightsquigarrow X$ such that $\varphi\psi$ and $\psi\varphi$ are the identity maps on dense open sets.*

Note 10.5. In this case φ and ψ are dominant so we have an isomorphism $\mathbb{K}(X) \simeq \mathbb{K}(Y)$ of function fields.

Example 10.6. The Cremona transformation. This is a map $\varphi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ defined by $(x : y : z) \longmapsto (xy : yz : zx)$. This is defined everywhere except where any two of x, y, z are nonzero (intersection of the axes) and contracts lines into points. The image is $\mathbb{P}^2 - \{\text{axes}\} \cup \{\text{intersection of axes}\}$.
Note that $\varphi^2(x : y : z) = \varphi(xy : yz : xz) = (xy^2z : xyz^2 : x^2yz) = (x : y : z)$, hence $\varphi^{-1} = \varphi$ and φ is a birational map. Consider $\varphi^* : \mathbb{K}\left(\frac{x}{z}, \frac{y}{z}\right) \longrightarrow \mathbb{K}\left(\frac{x}{z}, \frac{y}{z}\right)$ given by $\left(\frac{x}{z}, \frac{y}{z}\right) \longmapsto \left(\frac{z}{x}, \frac{z}{y}\right)$ (note that we can write $(x : y : z) \longmapsto (1/x : 1/y : 1/z)$.)

10.2. **Products.** We aim to show that $\mathbb{P}^n_{x_0,\dots,x_n} \times \mathbb{P}^m_{y_0,\dots,y_m}$ is naturally a projective variety. Let $N = (m+1)(n+1)-1$ and coordinates on $\mathbb{P}^N$ be $\{z_{ij}\}_{i\in[0,n],j\in[0,m]}$. The Segre embedding is the map

$$\begin{array}{rcccccccc} S : \mathbb{P}^n_{x_0,\dots,x_n} \times \mathbb{P}^m_{y_0,\dots,y_m} & & & \longrightarrow & \mathbb{P}^N & & & & \\ ((x_0 : \dots : x_n), (y_0 : \dots : y_m)) & = & (\boldsymbol{x}, \boldsymbol{y}) & \longmapsto & \boldsymbol{x}^T\boldsymbol{y} & = & (x_iy_j) & = & (z_{ij}) \end{array}$$

Lemma 10.7. *The map S is an embedding, and that $\mathrm{im}(S)$ is a closed subset of $\mathbb{P}^N$ defined by $z_{ij}z_{rs} = z_{is}z_{rj}$ for all $i, r \in [0,n]$ and $j, s \in [0,m]$.*

Proof. $\mathrm{im}(S)$ is the set of rank 1 matrices. Conversely for each rank 1 matrix, there exists a unique factorisation $A = x^Ty$ up to scalar multiplication.
The rank 1 matrices are those with vanishing 2×2 determinants, this gives the equations above. □

Proposition 10.8. *There are regular maps $\mathbb{P}^n_{x_0,\dots,x_n} \times \mathbb{P}^m_{y_0,\dots,y_m} \longrightarrow \mathbb{P}^n$, given by $(\boldsymbol{x}, \boldsymbol{y}) \longmapsto \boldsymbol{x}$*

$$\boldsymbol{x}^T\boldsymbol{y} \quad = \quad \begin{pmatrix} z_{00} & z_{01} & \\ \vdots & \ddots & \\ z_{n0} & & \end{pmatrix} \quad \longmapsto \quad \begin{array}{c} \begin{pmatrix} z_{00} & \dots & z_{n0} \end{pmatrix} \\ = \dots = \\ \begin{pmatrix} z_{0m} & \dots & z_{nm} \end{pmatrix} \end{array}$$

10.3. **Compatibility with affine local picture.** Let $U = \mathbb{P}^n_{x_0,\dots,x_n} - V(x_0) \simeq \mathbb{A}^n$, $U = \mathbb{P}^n_{y_0,\dots,y_m} - V(y_0) \simeq \mathbb{A}^m$,

Lemma 10.9. *$U \times V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to $\mathbb{A}^n \times \mathbb{A}^m$.*

Proof. $U \times V$ is the affine piece of $\mathbb{P}^n \times \mathbb{P}^m$ when $z_{00} = x_0y_0 = 1$. Using this we can use the vanishing determinant equations to fill out the matrix below

$$\begin{pmatrix} 1 & z_{01} & \dots & z_{0m} \\ z_{10} & z_{10}z_{01} & \dots & z_{10}z_{0m} \\ \vdots & \vdots & \ddots & \\ z_{n0} & z_{n0}z_{01} & & z_{nm} \end{pmatrix} \implies \mathbb{K}[U \times V] \quad = \quad \mathbb{K}[z_{10}, \dots, z_{n0}, z_{01}, \dots, z_{0m}] \quad = \quad \mathbb{K}[\mathbb{A}^n \times \mathbb{A}^m]$$

□

11. Graphs and Blowing up

Proposition 11.1. *Let $X \subseteq \mathbb{P}^n, Y \subseteq \mathbb{P}^m$ be projective varieties (closed in $\mathbb{P}^n$ and $\mathbb{P}^m$ resp) Then $X \times Y$ is a projective variety.*

Proof. Projection is regular implies it is continuous so $X \times Y = X \times \mathbb{P}^n \cap \mathbb{P}^m \times Y$ is closed. The proof of lecture 4 showing products of affine varieties was irreducible works to show $X \times Y$ is irreducible. □

11.1. **Graphs.** Let $\varphi : X \longrightarrow Y$ be a morphism (regular map) of quasi-projective varieties. Define the graph of φ to be

$$\Gamma_\varphi \quad := \quad \{(x, \varphi(x) \mid x \in X\} \quad \subseteq \quad X \times Y$$

Let the diagonal $\Delta := \{(y, y) \mid y \in Y\} \subseteq Y \times Y$.

Proposition 11.2. *$\Gamma_\varphi \subseteq X \times Y$ is closed.*

Proof. By enlarging Y, we can assume $Y = \mathbb{P}^m$. Let $\tilde{\varphi} : X \times Y \stackrel{(\varphi,\mathrm{id})}{\longrightarrow} Y \times Y$. Exercise: this is regular. Therefore $\Gamma_\varphi = \tilde{\varphi}^{-1}(\Delta)$, and it suffices to show the following □

Lemma 11.3. *$\Delta \subset \mathbb{P}^n_{x_0,\dots,x_n} \times \mathbb{P}^n_{y_0,\dots,y_n} \subseteq \mathbb{P}^N_{z_{00},\dots,z_{nn}}$ (via Segre embedding) is closed, In fact, Δ given by $x_iy_j = z_{ij} = z_{ji} = x_jy_i$.*

Proof. It is clear that points of Δ satisfy $z_{ij} = z_{ji}$. Conversely, for $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{P}^n \times \mathbb{P}^n$, then $\boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{y}^T\boldsymbol{x}$ by symmetry, we get $y \| x$, so the lemma is proved. □

11.2. **Closed subsets of $\mathbb{P}^n_{x_0,\dots,x_n} \times \mathbb{P}^m_{y_0,\dots,y_m}$.**

Definition 11.4. *$f \in \mathbb{K}[x_0,\dots,x_n,y_0,\dots,y_m]$ is bihomogeneous of degree (d,e) if it is homogeneous in x_i's of degree d and homogeneous in y_i's of degree e.*

Note 11.5. Zeroes of bihomogeneous polynomials in $\mathbb{P}^n \times \mathbb{P}^m$ are well defined.

Proposition 11.6. *The closed subsets of $S := \mathbb{P}^n_{x_0,\dots,x_n} \times \mathbb{P}^m_{y_0,\dots,y_m}$ are precisely the zeroes of bihomogeneous polynomials.*

Proof. Consider the Segre embedding $S \subseteq \mathbb{P}^N_{z_{00},\dots,x_{mn}}$. Let $f \in \mathbb{K}[z_{00},\dots,z_{mn}]_d$, then $V(f)\cap S$ are zeroes of bihomogeneous polynomials $f(x_0y_0,\dots,x_my_n)$ of degree (d,d). Conversely, let $f(x_0,\dots,x_n,y_0,\dots,y_m)$ be bihomogeneous of degree (d,e) (w.l.o.g. $d \geqslant e$). We have

$$\text{zeroes of } f \quad = \quad \text{common zeroes of } \{fy_0^{d-e},\dots,fy_m^{d-e}\} \text{ (each of degree } (d,d))$$

These in turn are homogeneous polynomials in $z_{ij} = x_iy_j$. □

Remark 11.7. Similarly closed subsets of $\mathbb{A}^n_{x_1,\dots,x_n} \times \mathbb{P}^m_{y_0,\dots,y_m}$, given by zeroes of polynomials $f(x_1,\dots,x_n,y_0,\dots,y_m)$ which are homogeneous in y_j's.

11.3. **Projections.** Let $E \subseteq \mathbb{P}^n_{x_0,\dots,x_n}$ be an r-plane defined by $V(l_1,\dots,l_{n-r}), l_j \in \mathbb{K}[x_0,\dots,x_n]_1$, or linear forms in $x_0,\dots,x_n$. Define a projection away from E to be the rational map

$$(l_1,\dots,l_{n-r}) : \mathbb{P}^n \quad \rightsquigarrow \quad \mathbb{P}^{n-r-1}$$

clearly this is not defined on E, and is regular on $\mathbb{P}^n - E$.

Example 11.8. Let $(1:0:0) = E \subseteq \mathbb{P}^2_{x,y,z} \rightsquigarrow \mathbb{P}^1$ defined by $(1:y:z) \longmapsto (y:z)$ (slope y/z). The regular locus of the map is $\mathbb{A}^2_{y,z} = \mathbb{P}^2 - V(x)$.

11.4. **Blowing up $\mathbb{P}^n$.** Let $p = (1:0:\ldots:0) \in \mathbb{P}^n_{x_0,\dots,x_n}$, let $U = \mathbb{P}^n - p$, we have the projection

$$\begin{aligned} \pi : U &\longrightarrow \mathbb{P}^{n-1} \\ (x_0:\ldots:x_n) &\longmapsto (x_1:\ldots:x_n) \end{aligned}$$

Consider the graph $\Gamma_\pi \overset{\text{closed}}{\subseteq} U \times \mathbb{P}^{n-1} \overset{\text{open}}{\subseteq} \mathbb{P}^n \times \mathbb{P}^{n-1}$.

Definition 11.9. *The blow up of $\mathbb{P}^n$ at p is the closure of Γ_π in $\mathbb{P}^n \times \mathbb{P}^{n-1}$. Denote it $\mathrm{Bl}_p(\mathbb{P}^n)$.*

The projection restricts to regular maps

$$\begin{array}{ccccc} & \mathrm{Bl}_p(\mathbb{P}^n) & & \subseteq & \mathbb{P}^n \times \mathbb{P}^{n-1} \\ \swarrow_\sigma & & \searrow & & \\ \mathbb{P}^n & & \rightsquigarrow & \mathbb{P}^{n-1} & \end{array}$$

Example 11.10. Draw picture, see hand out

Theorem 11.11. *Consider $\mathrm{Bl}_p(\mathbb{P}^n) \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$, then*

(1) $\sigma : \mathrm{Bl}_p(\mathbb{P}^n) \longrightarrow \mathbb{P}^n$ is an isomorphism above $U := \mathbb{P}^n - p$, σ is birational.
(2) $\mathrm{Bl}_p(\mathbb{P}^n)$ is irreducible
(3) $\sigma^{-1}(p) = \{(p,q) \mid q \in \mathbb{P}^{n-1}\} \simeq \mathbb{P}^{n-1}$
(4) $\mathrm{Bl}_p(\mathbb{P}^n)$ is defined by the bihomogeneous polynomials $x_iy_j = x_jy_i$, where $i,j \geqslant 1$.
(5) The affine piece of $\mathrm{Bl}_p(\mathbb{P}^n)$ where $x_0y_i \neq 0$ is isomorphic to $\mathbb{A}^n$.

Proof. (1) The inverse to σ is given by a map $U \longrightarrow \mathrm{Bl}_p(\mathbb{P}^n)$, $u \longmapsto (u,\pi(u))$.
(2) True since dense open U is irreducible.
(3) We need $\sigma^{-1}(p) = p \times \mathbb{P}^{n-1}$. Pick $(y_1:\ldots:y_n) = (\alpha_1:\ldots:\alpha_n) \in \mathbb{P}^{n-1}$. Then

$$((1:\lambda\alpha_1:\ldots:\lambda\alpha_n),(\alpha_1:\ldots:\alpha_n)) \quad \in \quad \Gamma_\pi$$

where $\lambda \in \mathbb{K}^* - \{0\}$. These are points of the affine line. Exercise: $\mathrm{Bl}_p(\mathbb{P}^n) = \overline{\Gamma}_\pi \ni ((1:0\ldots:0),(\alpha_1:\ldots:\alpha_n))$, where $(1:0:\ldots:0) = p$, so part 3 holds.
(4) Exercise: same as proof that $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$ defined by $x_iy_j = z_{ij} = z_{ji} = x_jy_i$.
(5) The affine piece where $x_0y_i \neq 0$ is isomorphic to $\mathbb{A}^n$, which is dense by dehomogenising homogeneous equations by the definition of blow up.

□

12. Completeness of projective varieties

Lemma 12.1. *Let $I \lhd \mathbb{K}[x_0, \dots, x_n]$ be homogeneous then $V(I) = \emptyset$ iff $I \supseteq \mathbb{K}[x_0, \dots, x_n]_{\geqslant s} =: \bigoplus_{d \geqslant s} \mathbb{K}[x_0, \dots, x_n]_d$ for some $s \in \mathbb{N}$.*

Proof. ($\Longleftarrow$) $V(I) \subseteq V(x_0^s, \dots, x_n^s) = \emptyset$. ($\Longrightarrow$) Let U' be the common zeroes of I in $\mathbb{A}^{n+1}_{x_0,\dots,x_n}$. Then $V(I) = \emptyset$ implies $I = \mathbb{K}[x_0, \dots, x_n]$ i.e. $U' = \{(0 : \dots : 0)\}$. By Hilbert's Nullstensatz, $\sqrt{I} = \mathcal{I}(V(I)) = \mathcal{I}(\emptyset) = (x_0, \dots, x_n)$, therefore for some $r \in \mathbb{N}$, $I \ni x_0^r, \dots, x_n^r$. Therefore $I \supseteq \mathbb{K}[x_0, \dots, x_n]_{\geqslant n(r+1)}$, so the lemma is proved. □

The rest of the lecture will be spent on proving the following

Theorem 12.2. *Consider the morphism $\varphi : X \longrightarrow Y$, where X is a projective variety and Y is a quasi-projective variety. Then $\operatorname{im}(\varphi) \subseteq Y$ is closed.*

The proof will follow from a sequence of lemmas. The graph $\Gamma_\varphi := \{(x, \varphi(x))\} \subseteq X \times Y$ is closed from last lecture. Since $\operatorname{im}(\varphi) = \operatorname{im}(\pi : \Gamma_\varphi \longrightarrow Y)$ (π is projection onto the second coordinate) it suffices to show

Lemma 12.3. *The map π is a closed map, i.e. takes closed sets to closed sets.*

Note 12.4. The above is not true if X is an affine variety.

Suppose $X \subseteq \mathbb{P}^n$ is closed and $Z \subseteq X \times Y$ is also closed. Since

$$\begin{array}{ccccc} Z & \subseteq & X \times Y & \xrightarrow{\pi} & Y \\ & & {\scriptstyle\iota}\downarrow & \nearrow_{\tilde{\pi}} & \\ & & \mathbb{P}^n \times Y & & \end{array}$$

commutes and ι is a closed map, it suffices to show $\tilde{\pi}$ is closed. We will need the following topological

Lemma 12.5. *Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of Y. Then $V \subseteq Y$ is closed iff $V \cap U_\alpha \subseteq U_\alpha$ is closed in the relative topology of U_α for all $\alpha \in A$.*

Proof. ($\Longrightarrow$) by definition. ($\Longleftarrow$) Let $Z_\alpha = Z \cup (Y - U_\alpha) = \overline{Z \cap U}_\alpha \cup (Y - U_\alpha)$ which is closed in Y. Then $Z = \bigcap_{\alpha \in A} Z_\alpha$. □

This implies we can replace Y with an element of any open cover of Y. Replacing Y with U where U is an affine piece of Y, it suffices to prove $\pi : \mathbb{P}^m \times \mathbb{A}^m \longrightarrow \mathbb{A}^m$ is a closed map, since U is closed in $\mathbb{A}^m$.

Theorem 12.6. *The projection map $\pi : \mathbb{P}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^m$ is closed.*

Proof. Let $Z \subseteq \mathbb{P}^n_{x_0,\dots,x_n} \times \mathbb{A}^m_{y_1,\dots,y_m}$ be a closed subset defined by the ideal $I \lhd \mathbb{K}[x_0, \dots, x_n, y_1, \dots, y_m]$ which is homogeneous in $\{x_0, \dots, x_n\}$. By lemma 12.1, $(y_1, \dots, y_m) = (\alpha_1, \dots, \alpha_m) = \alpha$ is not in $\operatorname{im}(\pi)$ iff the ideal $I_\alpha := \{f(x_0, \dots, x_n, \alpha_1, \dots, \alpha_m) \mid f \in I\}$ is such that $I_\alpha \supseteq \mathbb{K}[x_0, \dots, x_n]_{\geqslant s}$ for some $s \in \mathbb{N}$. Define

$$W_s := \{\alpha \in \mathbb{A}^m \mid I_\alpha \supseteq \mathbb{K}[x_0, \dots, x_n]_{\geqslant s}\}$$

so W_s are points not in $\operatorname{im}(\pi)$, and hence $\operatorname{im}(\pi) = \mathbb{A}^m - \bigcup_{s \in \mathbb{N}} W_s$. It suffices now to show □

Lemma 12.7. *The set W_s is open.*

Proof. Consider $V_s = \mathbb{K}[x_0, \dots, x_n]_s$. This is a finite dimensional vector space so let $\dim(V_s) = n_s$. so let $\mathcal{B}_s$ be a basis of monomials for V_s. Suppose I is generated by $f_1, \dots, f_t$ where $f_i(x_0, \dots, x_n, \alpha_1, \dots, \alpha_m)$ are homogeneous in the x_i's and of degree $d_i \leqslant s$. For any $q_j \in \mathcal{B}_{s-d_i}$,

$$f_i(x_0, \dots, x_n, y_1, \dots, y_m) q_j \in \mathbb{K}[x_0, \dots, x_n]_s$$

Let the coordinate vectors (i.e. coefficients of the monomials) of those $f_i q_j$ be $a^j_{i1}, \dots, a^j_{in_i} \in \mathbb{K}[\alpha_1, \dots, \alpha_m]$ for $1 \leqslant j \leqslant n_{s-d_i}$.

$$I_\alpha \cap \mathbb{K}[x_0, \dots, x_n]_{=s} = \operatorname{col} \begin{pmatrix} A^1 \\ \vdots \\ A^t \end{pmatrix} := \operatorname{col}(A)$$

$$\text{where} \quad A^i = \begin{pmatrix} a^1_{i1} & \dots & a^1_{in_s} \\ \vdots & \ddots & \\ a^{n_{s-d_i}}_{i1} & \dots & a^{n_{s-d_i}}_{in_s} \end{pmatrix}$$

this is equal to $\mathbb{K}[x_0, \dots, x_n]_s$ iff A is full rank. This is an open condition, hence W_s is open. □

This completes the proof of theorem 12.2.

Corollary 12.8. *Let X be a projective variety. Any regular function $f : X \longrightarrow \mathbb{K}$ is constant.*

Proof. f induces a regular map $\varphi_f : X \longrightarrow \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. Theorem 12.2 implies $\mathrm{im}(\varphi_\varphi)$ is closed, but it is not $\mathbb{P}^1$ so it is a finite set of points. X irreducible implies it is connected and φ continuous implies $\mathrm{im}(\varphi)$ is a single point. Hence f is constant. □

13. Finite maps

Example 13.1. Let X be an affine variety in $\mathbb{A}^n_{x_1,\ldots,x_n}$, and $Y \subseteq_{\text{closed}} X \times \mathbb{A}^1_y$, then $\mathbb{K}[Y] \simeq \mathbb{K}[X][y]/(y^d + a_1 y^{d-1} + \ldots + a_d)$ for $d > 0, a_i \in \mathbb{K}[x]$.

The projection map $\varphi : Y \longrightarrow X$ is surjective and has finite fibres. Consider $(x_1, \ldots, x_n) = (\alpha_1, \ldots, \alpha_n) = \alpha \in X$, and $\varphi^{-1}(\alpha)$ contains the points $(x_1, \ldots, x_n, y) = (\alpha_1, \ldots, \alpha_n, y)$, where y is a root of

$$y^d + a_1(\alpha) y^{d-1} + \ldots + a_d(\alpha) \quad \in \quad \mathbb{K}[y]$$

so $\varphi^{-1}(\alpha)$ is nonempty and finite.

Proposition 13.2. *An homomorphism of commutative $\mathbb{K}$-algebras $\varphi^* : A \longrightarrow B$ is said to be finite if either of two following equivalent conditions hold,*

1. *B is a finitely generated A-module.*
2. *B is a finitely generated A-algebra and B is integral over $\varphi^* A$ in the sense that if any $b \in B$ satisfies a monic polynomial equation of the form*

$$b^d + a_1 b^{d-1} + \ldots + a_d \quad = \quad 0$$

with $d > 0, a_i \in \varphi^* A$.

A morphism of affine varieties $\varphi : X \longrightarrow Y$ is finite if $\varphi^* : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$ is finite.

Proof. Commutative algebra, see lecture notes on web. □

Remark 13.3.

1. $\mathbb{K}[Y]$ a finitely generated algebra over $\mathbb{K}$ implies $\mathbb{K}[Y]$ a finitely generated algebra over $\mathbb{K}[X]$.
2. Definition depends only on the map $\varphi^* A \longrightarrow B$ and $\varphi^* \mathbb{K}[Y] \longleftrightarrow \overline{\varphi(X)}$. So $X \longrightarrow Y$ is finite iff $X \longrightarrow \overline{\varphi(X)}$ is finite.

Proposition 13.4.

1. *Composites of finite maps are finite.*
2. *$\varphi : X \longrightarrow Y$ finite implies the fibres are finite.*

Proof. Proof of 1 as for fields. For part 2, let $R = \varphi^* \mathbb{K}[Y]$ and suppose $\mathbb{K}[X]$ generated by $b_1, \ldots, b_r$ over R. Then

$$R \subseteq R[b_1] \subseteq R[b_1, b_2] \subseteq \ldots \subseteq R[b_1, \ldots, b_r] = \mathbb{K}[Y]$$

now use induction and example 13.1 to show the fibres are finite. □

Example 13.5. Let $U = \mathbb{A}^1_x - 0 \longrightarrow \mathbb{A}^1_x$, then $\mathbb{K}[U] = \mathbb{K}[x, x^{-1}] \longleftarrow \mathbb{K}[\mathbb{A}^1] = \mathbb{K}[x]$. This is not a finite map.

Theorem 13.6. *Any finite map is closed.*

Proof. Omitted, see notes on web. □

Proposition 13.7. *Let X and Y be quasi-projective varieties. Let $\{V_\alpha\}$ be a cover of Y by affine open sets. A morphism $\varphi : X \longrightarrow Y$ is finite if $\varphi^{-1}(V_\alpha)$ is a affine open subset of X and $\varphi^{-1}(V_\alpha) \longrightarrow V_\alpha$ is a finite map of affine varieties.*
This definition is independent of the choice of open covers.

Proof. Omitted, using partition of unity. □

13.1. **Projections and Noether normalisation.**

Theorem 13.8. *Let X be a closed subset of $\mathbb{P}^n_{x_0,\ldots,x_n}$ and let E be an r-plane in $\mathbb{P}^n$ such that $E \cap X = \emptyset$. Let $\pi : X \longrightarrow \mathbb{P}^{n-r-1}$ be the projection away from E. Then π is a finite map.*

Proof. By induction we can assume E =point= $(1 : 0 : \ldots : 0)$ (exercise). If $X' \subseteq X$ is closed, then $X' \longrightarrow X$ is finite. (Reason, if X is affine then $\mathbb{K}[X] \longrightarrow \mathbb{K}[X'] \simeq \mathbb{K}[X]/\mathcal{I}(X')$ this is a finitely generated $\mathbb{K}[X]$-algebra...). We can reduce to the case where $X = V(f)$ where $f \in \mathbb{K}[x_0, \ldots, x_n]_d$.
Consider $\pi : X \overset{(x_1:\ldots:x_n)}{\longmapsto} \mathbb{P}^{n-1}_{x_1,\ldots,x_n}$, look above the affine piece $U \subseteq \mathbb{P}^{n-1}_{x_1,\ldots,x_n}$ where $x_n \neq 0$, so

$$U \quad \simeq \quad \mathbb{A}^1_{\frac{x_1}{x_n},\ldots,\frac{x_{n-1}}{x_n}}$$

The affine piece $\pi^{-1}(U) \cap X$ is the affine piece of X where $x_n \neq 0$. We need $\pi^{-1}(U) \cap X \longrightarrow U$ to be a finite map of affine varieties. The corresponding $\mathbb{K}$-algebra map is

$$\mathbb{K}[\pi^{-1}(U) \cap X] \quad \overset{\pi^*}{\longleftarrow} \quad \mathbb{K}[U] \quad = \quad \mathbb{K}\left[\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right]$$

The subset of $\mathbb{A}^1_{\frac{x_1}{x_n},\ldots,\frac{x_{n-1}}{x_n}}$ defined by equation

$$\frac{f(x_0, \ldots, x_n)}{x_n^d} \quad = \quad f\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, 1\right) \quad = \quad 0$$

Note that $E \cap X = \emptyset$, so $f(1 : 0 : \ldots : 0) \neq 0$. Therefore $\mathrm{coeff}(x_0^d) \neq 0$ so can scale so coefficient is 1. Thus

$$\mathbb{K}[\pi^{-1}(U) \cap X] \quad = \quad \frac{\mathbb{K}[U][x_0/x_n]}{f\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, 1\right)}$$

f is monic, so by example 13.1, this is finite. □

13.2. **Noether normalisation.**

Theorem 13.9.
1. *Let $X \subseteq_{\text{closed}} \mathbb{P}^n$. Then there exists some finite surjective morphism $X \longrightarrow \mathbb{P}^m$.*
2. *Let $X \subseteq_{\text{closed}} \mathbb{A}^n$. Then there exists some finite surjective morphism $X \longrightarrow \mathbb{A}^m$.*

Proof. If $X \neq \mathbb{P}^n$ then pick a point on $\mathbb{P}^n - X$ and project from it. Continue inductively. Since composite of finite maps is finite, we eventually obtain part 1. Part 2 is proved by applying this sequence of projections (need to choose these projections carefully) to $\overline{X} \subseteq \mathbb{P}^n$. □

14. Dimension theory I

Let $\mathbb{M}$ be a finitely generated field extension of $\mathbb{K}$. Then there are algebraically independent elements $x_1, \ldots, x_r$ with $\mathbb{M}/\mathbb{K}(x_1, \ldots, x_r)$, where algebraically independent means that $\mathbb{K}(x_1, \ldots, x_r)$ is the field of rational functions in $x_1, \ldots, x_r$ (no algebraic relations). The number r is independent of the choice of $x_1, \ldots, x_r$ and is called the transcendence degree of $\mathbb{M}/\mathbb{K}$, denoted

$$r \quad = \quad \mathrm{tr}.\deg_{\mathbb{K}} \mathbb{M}$$

Example 14.1. Consider a finite surjective morphism $\varphi : X \longrightarrow \mathbb{A}^m_{x_1,\ldots,x_m}$, X irreducible, then we have

$$\mathbb{K}[x_1, \ldots, x_m][b_1, \ldots, b_s] \quad = \quad \mathbb{K}[X] \quad \longleftarrow \quad \mathbb{K}[x_1, \ldots, x_m] \quad = \quad \mathbb{K}[\mathbb{A}^m]$$

where the $b_1, \ldots, b_s$ are integral over $x_1, \ldots, x_m$.

$$\mathbb{K}(X) \quad = \quad \mathbb{K}(x_1, \ldots, x_m)[b_1, \ldots, b_s]$$

note that $\mathbb{K}(X)$ is the smallest field containing $\mathbb{K}[X]$. Therefore $\mathrm{tr}.\deg_{\mathbb{K}} \mathbb{K}(X) = m$, this suggest the following

Definition 14.2. *Given a quasi-projective variety X, define the dimension of X to be*

$$\dim(X) \quad = \quad \mathrm{tr}.\deg_{\mathbb{K}} \mathbb{K}(X)$$

More generally, given X_i quasi-projective varieties,

$$\dim\left(\bigcup_{i=1}^n X_i\right) \quad = \quad \max_i \dim(X_i)$$

Remark 14.3. $\mathbb{K}(X)$ depends only on any dense open subset of X. So the same is true of $\dim(X)$.

Example 14.4. Consider finite surjective morphisms $X \overset{\varphi}{\longrightarrow} \mathbb{A}^m$ and $Y \overset{\psi}{\longrightarrow} \mathbb{A}^n$. Then the map $\varphi \times \psi : X \times Y \longrightarrow \mathbb{A}^{m+n}$ is finite (exercise). Hence $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Theorem 14.5. *Let Y be a quasi-projective variety and $X \subseteq_{\text{closed}} Y$,*

(1) then $\dim(X) \leqslant \dim(Y)$
(2) if $\dim(X) = \dim(Y)$*, then* $X = Y$
(3) Suppose Y *is a projective variety and* $X = V(f) \subset Y$ *(strict inclusion, i.e.* $f(Y) \neq 0$*), then* $\dim(X) = \dim(Y) - 1$.

Proof. We can assume Y is projective by taking closure in $\mathbb{P}^n$. Construct Noether normalisation using projections from points to get a finite surjective map $Y \longrightarrow \mathbb{P}^{\dim(Y)}$. Keep going to get a finite surjective map $X \longrightarrow \mathbb{P}^{\dim(X)}$ and so $\dim(X) \leqslant \dim(Y)$. This gives part 1.
Part 2 follows from 3 and 1. We need the Veronese embedding to show part 3.

Example 14.6. The 2-Veronese embedding of $\mathbb{P}^1$ is

$$\begin{array}{ccc} \mathbb{P}^1_{x_0,x_1} & \longrightarrow & \mathbb{P}^2 \\ (x_0 : x_1) & \longmapsto & (x_0^2 : x_0 x_1 : x_1^2) \end{array}$$

In general the d-th Veronese embedding of $\mathbb{P}^n$ is the regular map

$$\begin{array}{ccc} \mathbb{P}^n_{x_0,\dots,x_n} & \longrightarrow & \mathbb{P}^N \\ (x_0 : \dots : x_n) & \longmapsto & (m_0 : \dots : m_N) \end{array}$$

where $m_0, \dots, m_N$ are all the degree d monomials in $x_0, \dots, x_n$.

Suppose $f \in \mathbb{K}[x_0, \dots, x_n]_d$ and $X \subset_{\text{closed}} Y \subseteq_{\text{closed}} \mathbb{P}^n \overset{d\text{-th Veronese}}{\longrightarrow} \mathbb{P}^N_{m_0,\dots,m_N}$. Now f is a linear combination of the $m_0, \dots, m_N$, so we can change coordinates so that $Y \subseteq \mathbb{P}^N_{y_0,\dots,y_N}$ and $X = V(f)$ with $f = y_0$. Apply the Noether normalisation algorithm as before, but only projecting from points in the hyperplane $y_0 = 0$. So say we have a map from

$$\mathbb{P}^s \overset{(y_0:l_1:\dots:l_{s-1})}{\rightsquigarrow} \mathbb{P}^{s-1}$$

where the $l_1, \dots, l_{s-1}$ are linear. This terminates with $\text{im}(Y) = \mathbb{P}^s$ (under the sequence of projections) containing $y_0 = 0$, and $Y \neq V(y_0) \subseteq \mathbb{P}^s$ by assumption. So as Y is irreducible (exercise) $Y = \mathbb{P}^s$ also $X = V(y_0)$ so $\dim(X) = \dim(V(y_0)) = s - 1 = \dim(Y) - 1$. □

Definition 14.7. *Let* Y *be a quasi-projective variety and* $X \subseteq_{closed} Y$*. The* <u>*codimension*</u> *is defined as* $\text{codim}_Y X = \dim(Y) - \dim(X)$*. A* <u>*hypersurface*</u> *in* Y *is one of form* $X = V(f) \cap Y$*,* f *a nonzero homogeneous polynomial on* Y*.*

Corollary 14.8. *Every irreducible component of a hypersurface in* $\mathbb{P}^n$ *or* $\mathbb{A}^n$ *has codimension* 1*.*

Proof. Do projective case only. Let $X = V(f) \subset \mathbb{P}^n$ with $f \in \mathbb{K}[x_1, \dots, x_n]_d$, and we factorise f into irreducibles $f = f_1 \dots f_r$ and note that all f_i are homogeneous. It suffices to show $V(f_i)$ are irreducible by theorem 14.5 part 3. Exercise f_i irreducible $\Longrightarrow$*every* dehomogenisation $\tilde{f}_i$ is irred. therefore $V(\tilde{f}_i)$ is irreducible which implies $V(f_i)$ is irreducible. □

Theorem 14.9.
(1) Let $X \subseteq_{closed} \mathbb{A}^n_{x_1,\dots,x_n}$ *be such that every irreducible component has codimension* 1*. Then* $X = V(f)$ *for* $f \in \mathbb{K}[x_1, \dots, x_n]$*.*
(2) Let $X \subseteq_{closed} \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ *be such that every irreducible component has codimension* 1*. Then* $X = V(f)$ *where* f *is homogeneous in each set of variables for each* $\mathbb{P}^{n_k}$*.*

Proof. Do part 1 only. Suffice to assume that X is irreducible. Pick f such that $X \subseteq V(f)$ with $f \neq 0$. Factorise f into irreducibles $f = f_1 \dots f_r$ so $X \subseteq V(f_1) \cup \dots \cup V(f_r)$. X irreducible implies $X \subseteq V(f_i)$ for some i. Now $\dim(X) = \dim(V(f_i))$ so theorem 14.5 part 2 implies $X = V(f_i)$. □

15. Dimension theory II

We saw that if we intersect a projective variety X with hypersurface $V(f)$ with $f \neq 0$, the dimension reduces by 1. Induction gives the following

Corollary 15.1. *Let* $X \subseteq_{closed} \mathbb{P}^n_{x_0,\dots,x_n}$*, and* $f_1, \dots, f_r \in \mathbb{K}[x_0, \dots, x_n]$ *with* f_k *homogeneous, then* $\dim(X \cap V(f_1, \dots, f_r)) \geqslant \dim(X) - r$.

Definition 15.2. *A* <u>*curve*</u> *(resp.* <u>*surface*</u>*) is a* 1*-dimensional (resp.* 2*-dimensional) quasi-projective variety.*

Corollary 15.3. *Any two projective curves in* $\mathbb{P}^2$ *intersect.*

Proof. We saw in lecture 14 that any curve in $\mathbb{P}^2_{x,y,z}$ is a hypersurface $V(f)$ for some homogeneous polynomial $f(x, y, z)$. Therefore the result follows form corollary 15.1. □

Theorem 15.4. *Let* X *be a quasi-projective variety in* $\mathbb{P}^n_{x_0,\dots,x_n}$ *and* $f_1, \dots, f_r \in \mathbb{K}[x_0, \dots, x_n]$ *with* f_k *homogeneous, then every component of* $X \cap V(f_1, \dots, f_r)$ *has dimension* $\geqslant \dim(X) - r$*.*

Remark 15.5. $X \cap V(f_1, \dots, f_r)$ may be $\emptyset$.

Proof. Omitted, not too hard. □

Corollary 15.6. *Let $X, Y \subset \mathbb{P}^n$ be quasi-projective varieties of dimensions d and e. Then every component of $X \cap Y$ has dimension $\geqslant d + e - n$.*

Proof. By considering affine pieces of $\mathbb{P}^n$ containing components of $X \cap Y$, we can assume $X, Y \subseteq \mathbb{A}^n$. Let $\Delta \subseteq \mathbb{A}^n \times \mathbb{A}^n$ be the diagonal. Then $X \cap Y \simeq X \times Y \cap \Delta$, and since Δ is defined b n equations, we apply the theroem to $X \times Y$ with $\dim(X \times Y) = d + e$. □

Definition 15.7. *Let $I \lhd \mathbb{K}[x_0, \dots, x_n]$ be a homogeneous ideal corresponding to a projective variety $X = V(I)$. The* <u>*affine cone of X*</u> *is*

$$V(I) \quad \subseteq \quad \mathbb{A}^{n+1}_{x_0,\dots,x_n}$$

Remark 15.8. Think of conic sections.

Corollary 15.9. *Consider projective varieties $X, Y \subseteq \mathbb{P}^n$ of dimensions d and e. If $d + e \geqslant n$ then $X \cap Y \neq \emptyset$.*

Proof. Consider affine cones $\tilde{X}, \tilde{Y} \subseteq \mathbb{A}^{n+1}$ of X and Y. Exercise: locally $\tilde{X} - 0 \simeq X \times (\mathbb{A}^1 - 0)$ implies $\dim(\tilde{X}) = \dim(X) + 1$. Now apply corollary 15.6 to $\tilde{X} \cap \tilde{Y} \subseteq \mathbb{A}^{n+1}$ noting that $0 \in \tilde{X} \cap \tilde{Y}$. This gives a component containing 0 having dimension $\geqslant (d+1) + (e+1) - (n+1) = d + e - n + 1 \geqslant 1$ (since we assume $d + e \geqslant n$), giving a point in $X \cap Y$,. □

15.1. **Theorem on dimension of fibres.**

Theorem 15.10. *Let $\varphi : X \longrightarrow Y$ be a dominant regular map of quasi-projective varieties, then*

(1) any component F of a fibre has $\dim(F) \geqslant \dim(X) - \dim(Y)$.
(2) there is a dense open $U \subseteq Y$ such that for any $y \in U$,

$$\dim(\varphi^{-1}(y)) \quad = \quad \dim(X) - \dim(Y)$$

Proof. We can assume Y is affine by replacing with appropriate affine space.

(1) Consider the fibre $\varphi^{-1}(y)$ and let F be one of its components. We can assume that X is affine by picking an affine open intersecting F. Pick a nonzero $f_1 \in \mathbb{K}[Y]$ such that $f_1(y) = 0$. Note $\dim(V(f_1)) = \dim(Y) - 1$. Exercise: we can $f_2 \in \mathbb{K}[Y]$ such that $f_2(y) = 0$ but f_2 is not identically zero on any component of $V(f_1)$. Hence $\dim(V(f_1, f_2)) = \dim(Y) - 2$. Continue inductively to get $f_1, \dots, f_{\dim(Y)} \in \mathbb{K}[Y]$ such that $\dim(V(f_1, \dots, f_{\dim(Y)})) = 0$, therefore $V(f_1, \dots, f_{\dim(Y)})$ has Y as a component (see assignment 2). F is a component of $V(\varphi^* f_1, \dots, \varphi^* f_{\dim(Y)})$. So theorem 1 implies $\dim(F) \geqslant \dim(X) - \dim(Y)$.

(2) Since X is covered by affine opens, we can replace X by an affine open. Since φ is dominant, $\varphi^* : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$ is injective. Noether normalisation implies there is a finite surjective map $Y \longrightarrow \mathbb{A}^{\dim(Y)}$, so we can assume $Y = \mathbb{A}^d_{y_1,\dots,y_d}$ where $d = \dim(Y)$. Exercise: we can find $x_{d+1}, \dots, x_e \in \mathbb{K}[X]$ such that $e = \dim(X)$, $y_1, \dots, y_d, x_{d+1}, \dots, x_e$ are algebraically independent and $\mathbb{K}(X)$ is algebraic over $K := \mathbb{K}(y_1, \dots, y_d, x_{d+1}, \dots, x_e)$. Suppose $\mathbb{K}[X]$ is generated by $t_1, \dots, t_r$ over $\mathbb{K}[y_1, \dots, y_d, x_{d+1}, \dots, x_e]$. That $\mathbb{K}(X)/K$ is algebraic implies there are relations of the form

$$a_0 t^{n_i} + a_i t^{n_i - 1} + \ldots + a_{n_i} \quad = \quad 0 \tag{2}$$

where $a_i \in \mathbb{K}[y_1, \dots, y_d, x_{d+1}, \dots, x_e]$. Pick a dense open $U \subseteq Y$ be a set of points, y, where the polynomial $a_0 \in \mathbb{K}[y_1, \dots, y_d, x_{d+1}, \dots, x_e]$ is nonzero when evaluated at y. Then $\mathbb{K}[\varphi^{-1}(Y)] = \mathbb{K}[x_{d+1}, \dots, x_e, t_1, \dots, t_r]$, where the $t_1, \dots, t_r$ are still algebraic over $\mathbb{K}(x_{d+1}, \dots, x_e)$, since the equation (2) reduces to a non trivial relation for $y \in U$. Therefore $\dim(\varphi^{-1}(Y)) = e - d$ as desired. (The algebraic relation (2) is only nontrivial generically).

□

Corollary 15.11. *Let $\varphi : X \longrightarrow Y$ be a dominant map of quasi-projective varieties. Then $\operatorname{im}(\varphi)$ contains a dense open set.*

16. Lines on surfaces in $\mathbb{P}^3$

Definition 16.1. *A* <u>*degree d hypersurface*</u> *in $\mathbb{P}^n_{x_0,\dots,x_n}$ is a variety of the form $V(f)$ where $f \in \mathbb{K}[x_0, \dots, x_n]_d$.*

Note 16.2. Most cubic surfaces in $\mathbb{P}^3$ has a finite nonzero number of lines on it.

16.1. **Compactified space of degree d surfaces in $\mathbb{P}^3$.** Let $W = \mathbb{K}[x_0, \ldots, x_3]_d$, $\mathbb{P}(W)$ parameterises degree d surfaces in $\mathbb{P}^3$ and has some extra points, e.g. $x_0^3 x_1$. Let $\Gamma = G(2,4)$, which parameterises lines in $\mathbb{P}^3$.

Definition 16.3. *Define the incidence correspondence,* $\mathbb{P}(W) \times \Gamma \supset I := \{(f, \ell) \mid V(f) \supseteq \ell,\ i.e.\ f(\ell) = 0\}$.

Theorem 16.4. *$I \subseteq \mathbb{P}(W) \times \Gamma$ is closed, i.e. I is defined by polynomial equations, which are homogeneous in the coefficients of $f \in W$ and homogeneous in Plücker coordinates p_{ij} of $\ell \in \Gamma$.*

Proof. Let $\ell \in \Gamma$ be a line joining $\alpha = (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3)$ and $\beta = (\beta_0 : \ldots : \beta_3)$. Let $f \in W - 0$, ℓ is the set $s\alpha - t\beta$ for $s, t \in \mathbb{K}$. Therefore $f(\ell) = 0$ iff for all $s, t \in \mathbb{K}$,

$$f(s\alpha_0 - t\beta_0 : \ldots : s\alpha_3 - t\beta_3) = 0$$

Note that rank $\begin{pmatrix} \alpha_0 & \ldots & \alpha_3 \\ \beta_0 & \ldots & \beta_3 \end{pmatrix} = 2$ so any $\begin{pmatrix} t \\ s \end{pmatrix}$ can be written as a linear combination

$$\begin{pmatrix} t \\ s \end{pmatrix} = u_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} + \ldots + u_3 \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix}$$

(think of the u's as variables). Therefore

$$s\alpha_0 - t\beta_0 = u_0 \cdot 0 + u_1(\beta_0\alpha_1 - \alpha_0\beta_1) + u_2 p_{02} + u_3 p_{03}$$

where $p_{0k} = \beta_0\alpha_k - \alpha_0\beta_k$ (check). Hence $(f, \ell) \in I$ iff

$$f(-u_1 p_{01} - u_2 p_{02} - u_3 p_{03}, \ldots) = 0 \tag{3}$$

linear combinations of p_{ij}'s with coefficient u's.
Consider *l.h.s.* of (3) as a polynomial in u_j's , (3) holds for all u_j's iff all coefficients are zero. Coeffices are polynomlas in coefficiencts f_i of f and p_{ij}'s. These polynomials are linear in f_j's and degree d in p_{ij}'s. This proves the theorem. □

Theorem 16.5. *Let $X \subseteq_{closed} \mathbb{P}^n$ and $X = \bigcup_i X_i$ be the decomposition of X into irreducible components. Let Y be a projective variety. Let $\varphi : X \longrightarrow Y$ be a continuous surjective map such that $\varphi|_{X_i} : X_i \longrightarrow Y$ is regular. Suppose all fibres are irreducible and have the same dimension n. Then X is irreducible.*

Proof. Exercise: show there is a dense open subset $U \subsetneq X$ such that the dimension of the fibres of $\varphi|_{X_i} : X_i \longrightarrow Y$ are constant on U, which is $\dim(X_i) - \dim(Y)$ unless empty.
Pick $y \in U$. Since φ is surjective, we can find an i such that $\varphi^{-1}(y) \cap X_i \neq \emptyset$. Now $\varphi^{-1}(y)$ is irreducible so we can find X_i such that $\varphi^{-1}(y) \cap X_i = \varphi^{-1}(y) = \bigcup(\varphi^{-1}(Y) \cap X_i)$. Note that $\varphi : X_i \longrightarrow Y$ is dominant but the image of projective varieties are closed implies $\varphi|_{X_i} : X_i \longrightarrow Y$ is surjective. Also the dimension of fibres of $\varphi|_{X_i} \geqslant \dim(\varphi^{-1}(y) \cap X_i) = \dim(\varphi^{-1}(y)) = n$. But all fibres of $\varphi : X \longrightarrow Y$ are irreducible of $\dim(n)$. So X_i contains all fibres of φ i.e. $X_i - X$ is irreducible. □

16.2. **Analysis of projections.**

$$\begin{array}{ccccc} & & I & \subseteq & \mathbb{P}(W) \times \Gamma \\ & \pi_1 \swarrow & & \searrow \pi_2 & \\ \mathbb{P}(W) & & & & \Gamma \end{array}$$

We use π_2 to get information on I, then extract information about π_1.

16.2.1. *Fibres of π_2, $\pi_2^{-1}(\ell)$.* By symmetry we can assume $\ell = V(x_0, x_1)$, $(f, \ell) \in \pi_2^{-1}(\ell)$ iff $f(\ell) = 0$. That is, we can write

$$f = x_0 f_0 + x_1 f_1 \tag{4}$$

with $f_0, f_1 \in \mathbb{K}[x_0, \ldots, x_3]_d$. Let $\tilde{W} \leqslant W$ be subspaces of these. Hence $\pi_2^{-1}(\ell) \simeq \mathbb{P}(\tilde{W})$. This is irreducible, as is Γ so theorem 16.5 implies I is irreducible.

Note 16.6. Recall $\dim(\mathbb{K}[x_0, \ldots, x_n]_d) = \binom{n+d}{d}$.

The expression (4) is unique if we assume $f_1 \in [x_1, x_2, x_3]_{d-1}$. Now we specialise to $d = 3$.

$$\begin{aligned} \dim(\pi_2^{-1}(\ell)) &= \dim(\mathbb{P}(\tilde{W})) = \dim(\tilde{W}) - 1 = \dim(\mathbb{K}[x_0, x_2, x_3]_{d-1}) + \dim(\mathbb{K}[x_1, x_2, x_3]_{d-1}) - 1 \\ \text{for } d = 3 \ldots &= \binom{3+2}{2} + \binom{4}{2} - 1 = 15 \end{aligned}$$

The theorem on dimension of fibres imply $\dim(I) = \dim(\Gamma) + \dim(\pi_2^{-1}(\ell)) = (4-2)\cdot 2 + 15 = 19$. Check the dimension of $\mathbb{P}(W)$

$$\dim(\mathbb{P}(W)) \quad = \quad \dim(W) - 1 \quad = \quad \binom{3+3}{3} 1 - 1 \quad = \quad 20 - 1 \quad = \quad 19$$

16.2.2. *Fibres of* π_1, $\pi_1^{-1}(\ell)$. Exercise: the cubic surface $x_0 x_1 x_3 = x_3^3$ as a finite number of nonzero lines on it (restrict to affine pieces).

If $\operatorname{im}(\pi_1) =: X$ has dim < 19, then all the nonempty fibres have dimension $\geqslant \dim(I) - \dim(X) \geqslant 1$ contradiction. Therefore $\dim(X) = 19$ and π_1 is surjective and theorem on dimension of fibres imply that fibres of π_1 are zero dimensional, that is finite by assignment question.

In summary $\mathbb{P}(W) \xleftarrow{\pi_1} I \subseteq \mathbb{P}(W) \times \Gamma$, π_1 is surjective, on $U \subseteq \mathbb{P}(W)$ dense and open (theorem on dimension of fibres). Fibres are zero dimensional. We say the generic cubic has a finite nonzero number of lines on it.

17. Tangent spaces

Consider $V(I) =: X \subseteq_{\text{closed}} \mathbb{A}^n_{x_1,\dots,x_n}$ and let $p := (p_1, \dots, p_n) \in X$. Define for $f \in \mathbb{K}[x_1, \dots, x_n]$,

$$\mathrm{d}_p f \quad = \quad \frac{\partial f}{\partial x_1}(p)(x_1 - p_1) + \ldots + \frac{\partial f}{\partial x_n}(p)(x_n - p_n)$$

Proposition 17.1. *We define the tangent space of X at p to be the common zeroes of $\{\mathrm{d}_p f\}_{f \in \mathcal{I}(X)}$. Denote it $T_{X,p}$. It is a vector space with $\mathbf{0}$ at p. If $\mathcal{I}(X) = (f_1, \dots f_r)$ then $T_{X,p} = V(\mathrm{d}_p f_1, \dots, \mathrm{d}_p f_r)$.*

Proof. Let $f = \sum a_i f_i$ where $a_i \in \mathbb{K}[x_1, \dots, x_n]$.

$$\mathrm{d}_p f \quad = \quad \sum_{i,j} \left(\frac{\partial a_i}{\partial x_j}(p) f_i(p) + \frac{\partial f_i}{\partial x_j} a_i(p) \right) (x_i - p_i) \ \in \ (\mathrm{d}_p f_1, \dots, \mathrm{d}_p f_r)$$

$f_i(p) = 0$ since $f_i \in I$ and summing over j gives $\sum a_i(p) \mathrm{d}_p f_i$. □

Proposition 17.2. *Restricting $\mathrm{d}_p f$ to $T_{X,p}$ is a linear functional such that it is zero when $f \in I$. This induces a linear map*

$$\mathrm{d}_p : \mathbb{K}[X] \quad = \quad \mathbb{K}[x_1, \dots, x_n]/I \quad \longrightarrow \quad T^*_{X,p}$$

Example 17.3. Consider $y^2 = x^3$ in $\mathbb{A}^2_{x,y}$, $p = (\alpha, \beta)$ and $f = y^2 - x^3$, then

$$\begin{aligned} 0 \quad = \quad \mathrm{d}_p f \quad &= \quad \frac{\partial f}{\partial x}(p)(x - a) + \frac{\partial f}{\partial y}(y - \beta) \\ &= \quad 3\alpha^2(x - \alpha) + 2\beta(y - \beta) \end{aligned}$$

This is a line if $p \neq (0,0)$ and is a plane when $p = (0,0)$.

Theorem 17.4. *Let $X \subseteq_{closed} \mathbb{A}^n$ and $p \in X$ and $\mathfrak{m} \in \mathbb{K}[X]$ be the maximal ideal corresponding to p. Let $\mathcal{O}_{X,p}$ be the local ring of $\mathbb{K}[X]$ at p and $\mathfrak{m}_p := \mathfrak{m}\mathcal{O}_{X,p}$. Then d_p induces an isomorphism*

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \quad \longrightarrow \quad \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\mathrm{d}_p} T^*_{X,p}$$

In particular, the tangent space $T_{X,p}$ is independent of coordinates and is an affine open set containing p.

Note 17.5. Local ring, $\mathcal{O}_{X,p}$, of X =quasi-projective variety at $p \in X$.

$$\begin{aligned} \mathcal{O}_{X,p} \quad &= \quad \{f \in \mathbb{K}(X) \mid f \text{ regular at } p\} \\ &= \quad \bigcup_{p \in U} \mathbb{K}[U] \qquad U \text{ affine neighbourhood of } p \\ &= \quad \mathbb{K}[U]_{\mathfrak{m}} \qquad U \text{ affine neighbourhood of } p \end{aligned}$$

and $\mathfrak{m}$ =maximal ideal of $\mathbb{K}[U]$ corresponding to p. (Stalk of structure sheaf at p.)

Remark 17.6. Hence we can define the tangent space at p only any quasi-projective variety X by picking any affine open set containing p.

Theorem 17.7. *Let $X \subseteq \mathbb{P}^n$ be locally closed (i.e. X is open in $\overline{X}$). Let $p \in X$ and $X' \subseteq X$ be the union of all irreducible components containing p. Then*

$$\dim_{\mathbb{K}}(T_{X,p}) \quad \geqslant \quad \dim_p(X) \quad =: \quad \dim(X')$$

We say X is smooth at p if equality holds above and say that p is singular point or singularity otherwise.

Example 17.8. Tangent space of a hypersurface. Consider $V(f) \subseteq \mathbb{A}^n_{x_1,\dots,x_n}$ and $\dim(V(f)) = n - 1$. $T_{X,p}$ is $\mathrm{d}_p f = 0$, one of the following occur

- $\mathrm{d}_p f$ is not identically zero, so $\dim_{\mathbb{K}} T_{X,p} = n - 1$ too i.e. p is smooth.
- $\mathrm{d}_p f = 0$ so $T_{X,p} = \mathbb{A}^n$ and p is singular. This happens on the closed set $0 = f = \frac{\partial f}{\partial x_1} = \ldots = \frac{\partial f}{\partial x_n}$.

Theorem 17.9. *Let $X \subseteq \mathbb{P}^n$ be locally closed. The set U of points where $\dim_{\mathbb{K}} T_{X,p} = \dim(X)$ is open and dense.*

Reason: Assume $V(I) =: X \subseteq \mathbb{A}^n$, $I = (f_1, \dots, f_r)$ and assume X is irreducible. Consider the tangent fibre space

$$X_{x_1,\dots,x_n} \times \mathbb{A}^n_{y_1,\dots,y_n} \quad \supseteq \quad \mathcal{T} \quad = \quad \{(p,v) \mid p \in X, v \in T_{X,p}\}$$

Can check that $\mathcal{T}$ is closed and defined by $f_1, \dots, f_r$ in the x variables, and $\frac{\partial f_i}{\partial x_j}$ in the y variables. The points in U are the points where the fibre dimension of $\mathcal{T} \longrightarrow X$ is min (= $\dim(X)$ by theorem 17.4. Then use the theorem on dimension of fibres.

Proposition 17.10. *The intersection points of two or more irreducible components are singular.*

Proof. In case $X = V(fg) = V(f) \cup V(g) \subseteq \mathbb{A}^n$. Let $p \in V(f) \cap V(g)$, $\dim(X) = n-1$.

$$\mathrm{d}_p(fg) \quad = \quad g(p)\mathrm{d}_p(f) + (\mathrm{d}_p g)f(p) \quad = \quad 0$$

since $g(p) = f(p) = 0$. Hence $T_{X,p} = \mathbb{A}^n$, so p is singular. □

Corollary 17.11. *Any connected smooth variety is irreducible. In particular any smooth hypersurface in $\mathbb{P}^n$ is irreducible, if $n \geqslant 2$.*

Note 17.12. The last statement follows from the fact that any two hypersurfaces in $\mathbb{P}^n$ for $n \geqslant 2$ must intersect.

Example 17.13. Suppose $X = V(f_1, \dots, f_r) \subseteq_{\text{closed}} \mathbb{P}^n_{x_0,\dots,x_n}$ where $f_1, \dots, f_r$ are homogeneous polynomials generating $\mathcal{I}(X)$. Let $p \in X$, then the closure of $T_{X,p}$ in $\mathbb{P}^n$ is the zeroes of

$$\frac{\partial f_i}{\partial x_0}(p)x_0 + \dots + \frac{\partial f_i}{\partial x_n}(p)x_n \quad = \quad 0$$

for $i \in [1, r]$.

18. Smooth varieties and local equations

Let X be a quasi-projective variety.

Definition 18.1. *$Y \subseteq X$ is a* <u>*subvariety*</u> *if it is an irreducible closed subset.*

18.1. **Local equations.**

Definition 18.2. *Let $Y \subseteq X$ be a codimension 1 (w.r.t. X subvariety and $p \in Y$. A* <u>*local equation*</u> *for Y at p is some $f \in \mathbb{K}[U]$ where $U \subseteq X$ is a sufficiently small affine open neighbourhood of p such that $\mathcal{I}(Y \cap U) \lhd \mathbb{K}[U]$ is the principal ideal (f).*

Example 18.3. Consider $V(x,y,z) \in \mathbb{P}^2_{x,y,z}$ where $p = V(x) \cap V(y)$. At p, on affine open $U_z \subseteq \mathbb{P}^2$ where $z \neq 0$, $\mathbb{K}[U_z] = \mathbb{K}\left[\frac{x}{z}, \frac{y}{z}\right]$. The subvariety $X = V(x) \subseteq \mathbb{P}^2$ is given by $x/z = 0$ on U_z, therefore x/z gives local equation.

Remark 18.4. We saw (lecture 14 ??) that $\mathbb{K}[x_1, \dots, x_n]$ is a UFD implies any codimension 1 subvariety $Y \subseteq \mathbb{A}^n$ has the form $V(f)$. This holds locally in the smooth case.

Theorem 18.5. *Let Y be a codimension 1 subvariety of X. Let $p \in Y$ be a smooth point of X, then $\mathcal{O}_{X,p}$ is a UFD. Hence there is a local equation for Y at p.*

Example 18.6. Consider $X = V(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2_{x,y}$, then $p = (0,0)$ is singular. There is no local equation for p on X.

$$\mathbb{K}[X] \quad = \quad \frac{\mathbb{K}[x,y]}{(y^2 - x^2(x+1))} \quad \lhd \; \mathfrak{m}_p = (x,y)/(y^2 - x^2(x+1))$$

$$\begin{aligned} T^*_{X,p} \;\simeq\; \mathfrak{m}_p/\mathfrak{m}_p^2 \quad &= \quad (x,y)/(x^2, xy, y^2, y^2 - x^2(x+1)) \\ &= \quad (x,y)/(x^2, xy, y^2) \end{aligned}$$

is 2-dimensional over $\mathbb{K}$. Moreover $\mathfrak{m}_p$ is not principal, else $\mathfrak{m}_p/\mathfrak{m}_p^2$ is generated by 1 element, this is impossible since $\dim_{\mathbb{K}}(\mathfrak{m}_p/\mathfrak{m}_p^2) = 2$ and for $z \in \mathfrak{m}_p/\mathfrak{m}_p^2$, $\mathbb{K}[x]z = \mathbb{K}z$. In fact, this argument applies to any affine open neighbourhood of p, using more commutative algebra.

Example 18.7. Consider a smooth curve X and $p \in X$. Pick $f \in \mathbb{K}[X]$ with $\mathrm{d}_p f \neq 0$. We can replace f with $g := f - f(p)$ such that $g(p) = 0$, then $g = 0$ is a local equation for p.

Theorem 18.8. *Let $\varphi : X \rightsquigarrow \mathbb{P}^n$ be a rational map where X is a smooth variety. Then φ is defined on an open set whose complement has codimension $\geqslant 2$.*

Example 18.9. A rational map is give at $p \in X$ by $\varphi = (\varphi_0 : \ldots : \varphi_n)$ where $\varphi_i \in \mathbb{K}(X)$. Note that changing p we may have to replace $(\varphi_0 : \ldots : \varphi_n)$ with $(\psi\varphi_0 : \ldots : \psi\varphi_n)$ for some $\psi \in \mathbb{K}(X)$. The map φ is defined where some $\varphi_i \neq 0$ and all denominators of $\varphi_i \neq 0$. This is open. Therefore φ is defined on an open set. To show it is defined off a codimension 2 set, suppose, to the contrary, that it is not defined on a codimension 1 subvariety $Y \subseteq X$. Let $p \in Y$ and $g = 0$ be a local equation for Y at p. Suppose $\varphi = (\varphi_0 : \ldots : \varphi_n)$, by shrinking X to sufficiently small affine open neighbourhood we can assume

(1) All $\varphi_i \in \mathcal{O}_{X,p} \subseteq \mathbb{K}(X)$ (i.e. clear denominators), and
(2) there are no common factors in the φ_i's (considered as elements of $\mathcal{O}_{X,p}$, since $\mathcal{O}_{X,p}$ is a UFD.)

1. implies φ is defined where not all $\varphi_i = 0$. Also 2. implies g is not a factor of some φ_i i.e. φ_i is not identically 0 on Y. Therefore φ is defined on most points of Y, a contradiction.

Corollary 18.10. *Any rational map $\varphi : X \rightsquigarrow \mathbb{P}^n$ where X is a smooth curve is regular.*

Corollary 18.11. *Two birational smooth projective curves are isomorphic.*

Proposition 18.12. *Any smooth conic (i.e. degree 2 projective curve) in $\mathbb{P}^2$ is isomorphic in $\mathbb{P}^1$.*

Proof. Let $q \in \mathbb{K}[x, y, z]_2$ and $\varphi : V(q) = X \rightsquigarrow \mathbb{P}^1_{s,t}$ be projection away from p ($p' \in X$ is sent to the line through p' and p and $p \in X$ is sent to the tangent line at p). By corollary 18.10, this is regular since X is smooth. We will show the inverse ψ is rational as well and is a regular inverse.
Change coordinates so that $p = (0 : 0 : 1)$ for simplicity assume $q(x, y, z) = xz - y^2$. A line through p has form $sx + ty = 0$. Intersect this with $V(q)$, and scale $z = 1$

$$0 = x - \left(-\frac{s}{t}x\right)^2 = x\left(1 - \frac{s^2}{t^2}x\right)^2$$

there are two roots, $x = 0$ (which is p) and $x = t^2/s^2$ (this is the x value of $\psi(s : t)$). The y value of $\psi(s : t)$ is $y = -t/s$, so $\psi(s : t) = \left(\frac{t^2}{s^2} : -\frac{t}{s} : 1\right)$ is rational and ψ is a regular inverse. $\square$

19. Divisors

The aim is to study subvarieties of a quasi-projective variety X and incidences between them. An easy case are codimension 1 subvarieties.

Definition 19.1. *A prime divisor of X is a codimension 1 subvariety. A divisor is a formal finite sum of the form*

$$D = \sum_{i=1}^{k} n_i H_i$$

where H_i are prime divisors, $n_i \in \mathbb{Z}$. Let $\mathrm{Div}(M)$ be the free abelian group generated by prime divisors, called the divisor group. The support of D is $\mathrm{supp}(D) = \bigcup D_i$. A divisor is effective, denoted $D \geqslant 0$ if all $n_i > 0$. Finally if $D - D' \geqslant 0$, then we write $D \geqslant D'$.

19.1. **Valuations.** Assume X is a smooth variety and D be a prime divisor, we wish to define

$$\begin{aligned} \nu_D : \mathbb{K}(X)^* &\longrightarrow \mathbb{Z} \\ f &\longmapsto \text{“order of zero or pole along } D\text{”} \end{aligned}$$

Restrict to affine open neighbourhood U of $p \in D$ such that D has local equation π on U (uses X smooth), that is

$$\mathcal{I}(D \cap U) = (\pi) \lhd \mathbb{K}[U]$$

Let $f = g/h \in \mathbb{K}(X)^*$ with $g, h \in \mathbb{K}[U] \subset \mathcal{O}_{X,p}$. Now $\mathcal{O}_{X,p}$ is a UFD so we can write $g = \pi^d \tilde{g}$ uniquely, where $\tilde{g} \in \mathcal{O}_{X,p} - \pi\mathcal{O}_{X,p}$ is coprime to π in $\mathcal{O}_{X,p}$. Define $\nu_D(g) = d$ and $\nu_D(f) = \nu_D(g) - \nu_D(h)$.

Proposition 19.2.
(1) $\nu_D : \mathbb{K}(X)^* \longrightarrow \mathbb{Z}$ well defined in the sense that it is independent of how you write $f = g/h$ and p and U.
(2) Can compute $d = \nu_D(g)$ in $\mathbb{K}[U]$ as the $d \in \mathbb{Z}$ such that $g \in \pi^d\mathbb{K}[U] - \pi^{d+1}\mathbb{K}[U]$.

Example 19.3. Consider $X = \mathbb{P}^2_{x,y,z}$, $f = \frac{xy - z^2}{x^2} \in \mathbb{K}(X) = \mathbb{K}(x, y, z)$. Look at the affine open $U \subseteq \mathbb{P}^2$ where $y \neq 0$, then $\mathbb{K}[U] = \mathbb{K}\left[\frac{x}{y}, \frac{z}{y}\right]$. For $D = V(x)$ on U has local equation $\pi = x/y$, therefore

$$\begin{aligned} \nu_D(f) &= \nu_D\left(\frac{xy - z^2}{y^2}\right) - \nu_D\left(\frac{x^2}{y^2}\right) \\ &= \nu_D\left(\pi - \frac{z^2}{y^2}\right) - \theta_D(\pi^2) = 0 - 2 = -2 \end{aligned}$$

since $\pi - \frac{z^2}{y^2}$ is coprime to π. Similarly if $D = V(xy - z^2)$, then local equation is ...

$$\begin{aligned} \nu_D(f) &= \nu_D\left(\frac{xy - z^2}{y^2}\right) - \nu_D\left(\frac{x^2}{y^2}\right) \\ &= 1 - 0 \ = \ 1 \end{aligned}$$

and $\nu_D(f) = 0$ for any other D.

19.2. **Principal divisors.** Let X be a smooth variety

Proposition 19.4. *The map*

$$\begin{aligned} \mathrm{div} : \mathbb{K}(X)^* &\longrightarrow \mathrm{Div}(X) \\ f &\longmapsto \sum_{D \text{ prime divisor}} \nu_D(f) D \end{aligned}$$

is a well defined group homomorphism. The image consists of <u>*principal divisors*</u>*.*

The zeroes of regular functions have finite number of irreducible components so only finitely many $\nu_D(f) \neq 0$. Hence this is well defined. Each ν_0 is additive so div is a group homomorphism.

Example 19.5. Let $X = \mathbb{P}^n_{x_0,\dots,x_n}$ and $f = \frac{g_1^{m_1} \dots g_r^{m_r}}{h_1^{n_1} \dots h_s^{n_s}} \in \mathbb{K}(X)$, where g_i and h_i are distinct irreducible homogeneous polynomials in $x_0, \dots, x_n$. Then

$$\mathrm{div}(f) = \sum m_i V(g_i) - \sum n_j V(h_j)$$

Note that $\sum m_i \deg(g_i) = \sum n_i \deg(h_i)$.

Definition 19.6. *The quotient* $\mathrm{Div}(X)/\mathrm{div}(\mathbb{K}(X)^*)$ *is called the* <u>*Picard group*</u> *denoted* $\mathrm{Pic}(X)$ *or the* <u>*class group*</u> *denoted* $\mathrm{Cl}(X)$*. The elements in* $\mathrm{Cl}(X)$ *are called divisor classes. Given* $D, D' \in \mathrm{Div}(X)$ *in same divisor class, we write* $D \sim D'$ *and say that* D *is* <u>*linearly equivalent*</u> *to* D'*.*

Remark 19.7. We think of linearly equivalent divisors as being continuous deformations of each other. For example $X = \mathbb{P}^2_{x,y,z}$, $f = \frac{z^2 - xy}{xy}$ so $\mathrm{div}(f) = V(z^2 - xy) - V(x) - V(y)$, and $V(z^2 - xy) \sim V(x) + V(y)$. On the affine open $z \neq 0$,

$$\begin{array}{lll} f = 0 & XY = 1 & \\ f = 1 & XY = \frac{1}{2} & \\ \vdots & & \\ f = \infty & X = 0 & Y = 0 \end{array}$$

draw diagram to show continuous deformation of hyperbolae approaching the axes.

Proposition 19.8. *We have a group homomorphism*

$$\begin{aligned} \mathrm{Pic}(\mathbb{P}^n_{x_0,\dots,x_n}) &\xrightarrow{\sim} \mathbb{Z} \\ \mathrm{Div}(\mathbb{P}^n) \ni V(f) &\longmapsto d \end{aligned}$$

where $f \in \mathbb{K}[x_0, \dots, x_n]_d$ *is irreducible. More generally*

$$\begin{aligned} \mathrm{Pic}(\underbrace{\mathbb{P}^n \times \dots \times \mathbb{P}^n}_{r \text{ times}}) &\xrightarrow{\sim} \mathbb{Z}^r \\ V(f) &\longmapsto (d_1, \dots, d_r) \end{aligned}$$

where f *is multi-homogeneous of multi-degree* $(d_1, \dots, d_r)$ *and irreducible.*

Proof. Comments. Saw in lecture 14 that any prime divisor D has form $D = V(f)$ so have defined $\deg : \mathrm{Div}(\mathbb{P}^n) \longrightarrow \mathbb{Z}$. This has kernel the principal divisors by example 19.5. □

20. Intersecting divisors and curves

Let $\varphi : Y \longrightarrow X$ be a regular map of smooth varieties and $D \in \mathrm{Div}(X)$. We wish to define $\varphi^* D \in \mathrm{Div}(Y)$. It suffices to defined φ^* for prime divisors and extend linearly. We need to assume $\varphi(Y) \nsubseteq \mathrm{supp}(D)$. To define $\varphi^* D$ it suffices to describe it on an open cover of Y. Let $\{U_i\}$ be an affine open cover of X such that on U_i, D has local equations $\pi_i \in \mathbb{K}[U_i]$. let $\{V_{ij}\}_j$ be an affine open cover of $\varphi^{-1}(U_i) \subseteq Y$ and

$$\varphi_{ij} = \varphi|_{V_{ij}} : V_{ij} \longrightarrow U_i$$

Note that if $U_i \cap \mathrm{supp}(D) = \emptyset$, then we can take $\pi_i = 1$. Consider $\varphi_{ij}^* \pi_i \in \mathbb{K}[V_{ij}]$ and note that since $\varphi(Y) \nsubseteq \mathrm{supp}(D)$, so $\varphi_{ij}^* \pi_i \neq 0$. Therefore we can define $\varphi^* D$ on V_{ij} to be $\mathrm{div}(\varphi_{ij}^* \pi_i)$.

Proposition 20.1.

1. *This definition is independent of* U_i*'s,* V_{ij}*'s, and* π_i*'s.*
2. $\mathrm{supp}(\varphi^* D) \subseteq \varphi^{-1}\,\mathrm{supp}(D)$*, since* $\mathrm{div}(\pi^* 1) = \mathrm{div}(1) = 0 \in \mathrm{Div}(Y)$*.*

Example 20.2. Consider $\mathbb{P}^1_{y,z} \simeq V(x) = Y \overset{\varphi}{\hookrightarrow} X = \mathbb{P}^2_{x,y,z}$. Let $D = V(yz^2 - x^3 + xz^2) \subseteq \mathrm{Div}(X)$. find $\varphi^* D \in \mathrm{Div}(Y)$ (Picture: D is a cubic intersecting $V(x)$ at $(0:0:1)$ and tangent to $V(x)$ at $(0:1:0)$). Let U be the affine piece where $z \neq 0$, here D has local equation

$$\begin{aligned} \pi &= \frac{y}{z} - \left(\frac{x}{z}\right)^3 + \frac{x}{z} \\ \mathbb{K}\left[\frac{y}{z}\right] \ni \varphi^*\pi &= \frac{y}{z} \\ \mathrm{div}\left(\frac{y}{z}\right) &= (0:0:1) \text{ on } U \end{aligned}$$

since $y/z = 0, z \neq 0$ which we scale to 1 and $x = 0$. On V, the affine piece where $y \neq 0$, D has local equation

$$\begin{aligned} \pi &= \left(\frac{z}{y}\right)^2 - \left(\frac{x}{y}\right)^3 + \frac{xz^2}{y^3} \\ \varphi^*\pi &= \left(\frac{z}{y}\right)^2 \in \mathbb{K}\left[\frac{z}{y}\right] = \mathbb{K}[Y - (0:0:1)] \\ \mathrm{div}\,(\varphi^*\pi) &= 2(0:1:0) \end{aligned}$$

therefore on all of Y $\varphi^* D = 2(0:1:0) + (0:0:1)$.

Proposition 20.3. *(Chow's moving lemma) Let X be a smooth variety and $D \in \mathrm{Div}(X)$, $x_1, \ldots, x_r \in X$. Then there exists $D' \in \mathrm{Div}(X)$ with $D' \sim D$ and no $x_i \in \mathrm{supp}(D')$.*

There is a group homomorphism $\varphi^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$. Suppose $D \in \mathrm{Div}(X)$ and $[D]$ is its divisor class. Pick any $D' \sim D$ such that $\varphi(Y) \nsubseteq \mathrm{supp}(D')$ (exists by Chow's moving lemma). Then $\varphi^*[D] = [\varphi^* D]$. This is functorial.

Proposition 20.4. *Let X be a smooth variety, D be a prime divisor and $U = X - D$. Consider $\varphi : U \hookrightarrow X$ be the inclusion map, then $\varphi^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(U)$ is surjective. Its kernel is the subgroup generated by $[D]$. In particular*

$$\begin{aligned} \mathrm{Pic}(\mathbb{A}^n) &= \mathrm{Pic}(\mathbb{P}^n_{x_0,\ldots,x_n} - V(x_0)) \\ &= \mathbb{Z}/\mathbb{Z} = 0 \\ \mathrm{Pic}(\mathbb{P}^2 - \textit{irred conic}) &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

Proof. (sketch) In this case $\varphi^* : \mathrm{Div}(X) \longrightarrow \mathrm{Div}(U)$ is always defined and surjective to compute the kernel note that $\mathbb{K}(X) = \mathbb{K}(U)$. Computing $\mathrm{div}(f)$ for $f \in \mathbb{K}(X) = \mathbb{K}(U)$ differs in X and U only in coefficient of D. □

20.1. **Degree of divisors.**

Proposition 20.5. *Let X be a smooth projective curve, define*

$$\begin{aligned} \deg : \mathrm{Div}(X) &\longrightarrow \mathbb{Z} \\ \sum_{p \in X} n_p p &\longmapsto \sum n_p \end{aligned}$$

Then the kernel of deg *contains all principal divisors so we get an induced group homomorphism* $\deg : \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$.

Heuristic reason: continuously deforming divisors does not change deg and the image of deg is discrete.

20.2. **Intersection numbers.** Let X be a smooth projective variety, C be a smooth projective curve in X. The inclusion $\iota : C \hookrightarrow X$. Let $D \in \mathrm{Div}(X)$ with divisor class $[D]$. Define intersection numbers $C \cdot D := C \cdot [D] := \deg(\iota^*[D]) \in \mathrm{Div}(C) \in \mathbb{Z}$. If $D \geqslant 0$ and $\iota^* D = n_1 p_1 + \ldots + n_r p_r$ we say that D intersects C at p_i with intersection multiplicity n_i. If D is prime then $n_i \geqslant 2$ means D is tangent to C at p_i.

Example 20.6. Let $X = \mathbb{P}^n_{x_0,\ldots,x_n}$, and $H = V(x_0)$ which generates $\mathrm{Pic}(\mathbb{P}^n)$. Let ℓ be the line $0 = x_0 = \ldots = x_{n-2}$ in H. Find $H \cdot \ell$: move H to $H' = V(x_n) \sim H$. Compute $H' \cdot \ell$ to be 1.

Note that since $D \geqslant 0$ implies $\iota^* D \geqslant 0$ (if $\iota^* D$ is defined, if not defined, this can be false) so $C \cdot D \geqslant 0$.

21. Intersecting divisors

Let X be a smooth projective variety of dimension n. Let $D_1, \dots, D_n \in \mathrm{Div}(X)$

Definition 21.1. *We say that $D_1, \dots, D_n$ are in **general position** if* $\dim \left(\bigcap_{i=1}^n D_i\right) = 0$.

Aim: to define intersection number $D_1 \dots D_n$ in this case. By extending linearly, it suffices to define when $D_1, \dots, D_n$ are prime divisors. Note that by assignment q2 $\bigcap \mathrm{supp}(D_i)$ is just a finite set of points. Let $p \in X$, we can pick a sufficiently small affine open neighbourhood, U of p such that

a) the only point of $\bigcap \mathrm{supp}(D_i)$ in U is p.
b) D_i is defined by a local equation $\pi_i \in \mathbb{K}[U]$ for all i.

Note that $\{p\} = V(\pi_1, \dots, \pi_n) \supseteq U$ and by the Nullstellensatz, $(\pi_1, \dots, \pi_n) \supseteq \mathfrak{m}^r$ for some $r \in \mathbb{N}$ where $\mathfrak{m} \lhd \mathbb{K}[U]$ is the maximal ideal corresponding to p.
Exercise: $\dim_{\mathbb{K}}(\mathbb{K}[U]/\mathfrak{m}^r) < \infty$ (clear for $\mathbb{K}[U]$ =polynomial ring). Hence we can make

Proposition 21.2. *The **local intersection number***

$$(D_1 \dots D_n)_p \quad := \quad \dim \left(\frac{\mathbb{K}[U]}{\mathbb{K}(\pi_1, \dots, \pi_n)} \right) \quad < \quad \infty$$

This is independent of the choice of U and π_i's (in fact, we can replace $\mathbb{K}[U]$ with $\mathcal{O}_{X,p}$.)

Some points

- If $n = 2$, and D_1 is a smooth curve then $(D_1 D_2)_p$ is the multiplicity of D_1 intersecting D_2
- If $p \notin \mathrm{supp}(D_i)$, the one can take $\pi_i = 1$,

$$(D_1 \dots D_n)_p \quad = \quad \dim_{\mathbb{K}} \left(\frac{\mathbb{K}[U]}{(\dots, 1, \dots)} \right) \quad = \quad 0$$

Also if $p \in \bigcap \mathrm{supp}(D_i)$, then $(D_1 \dots D_n)_p \geqslant 1$ since $\mathfrak{m} \supset (\pi_1, \dots, \pi_n)$ (exercise).

Definition 21.3. *The intersection number is defined as*

$$D_1 \dots D_n \quad = \quad \sum_{p \in X} (D_1 \dots D_n)_p \quad = \quad \sum_{p \in \bigcap \mathrm{supp}(D_i)} (D_1 \dots D_n)_p$$

on prime divisors $D_1, \dots, D_n$ and extend via linearity in each D_i to all divisors.

If $D_1, \dots, D_n \geqslant 0$ and in general position, then $D_1 \dots D_n \geqslant 0$.

Example 21.4. Let $X = \mathbb{P}^n_{x_0, \dots, x_n}$, $D_i = V(x_i)$, $i = 1, \dots, n$. Noe that $\bigcap \mathrm{supp}(D_i) = \{p = (1 : 0 : \dots : 0)\}$. Consider affine open neighbourhood $U \subset \mathbb{P}^2$ where $x_0 \neq 0$ so $\mathbb{K}[U] = \mathbb{K}\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right]$, the D_i is given by local equations $\pi_i = x_i/x_0 \in \mathbb{K}[U]$,

$$D_1 \dots D_n \quad = \quad (D_1 \dots D_n)_p \quad = \quad \dim_{\mathbb{K}} \left(\frac{\mathbb{K}\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right]}{\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)} \right) \quad = \quad 1$$

21.1. **General definition of intersecting divisors.** Let X be a smooth projective variety of dimension n. There is a well defined map

$$\begin{aligned} \cap : \mathrm{Pic}(X)^n &\longrightarrow \mathbb{Z} \\ ([D_1], \dots, [D_n]) &\longmapsto [D_1] \cdot \dots \cdot [D_n] \end{aligned}$$

where $[D_i]$ is the divisor class of some $D_i \in \mathrm{Pic}(X)$. The map $\cap$ is linear ineach copy of $\mathrm{Pic}(X)$ and symmetric in these variables. It can be characterised as follows; move D_i to linearly equivalent divisor so that they are in general position. This is possible by Chow's moving lemma. Then define $[D_1] \dots [D_n] := D_1 \dots D_n$. To prove this, we need to show that the intersection number is invariant under changing D_i to D_i' where $D_i' \sim D_i$. We have the following corollary

Theorem 21.5. *(Bézout)*

(1) Let $X = \mathbb{P}^n$, $D_1, \dots, D_n$ be irreducible hypersurfaces of degree d_i. Recall $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$, then

$$\begin{array}{ccccc} \mathrm{Pic}(X)^n & \xrightarrow{c_1^n} & \mathbb{Z}^n & \longrightarrow & \mathbb{Z} \\ ([D_1], \dots, [D_n]) & \longmapsto & (d_1, \dots, d_n) & \longmapsto & d_1 \dots d_n \end{array}$$

In particular, if $D_1, \dots, D_n$ are in general position, then no points in $\bigcap \mathrm{supp}(D_i) \leqslant d_1 \dots d_n$. Further if $n = 2$ and D_1 and D_2 are smooth curves not tangent to each other then we have equality, $\bigcap \mathrm{supp}(D_i) = d_1 \dots d_n$.

(2) Let $X = \mathbb{P}^m \times \mathbb{P}^n$, $D_1, \dots, D_{m+n}$ be divisors of bidegree $(d_1, e_1), \dots, (d_{m+n}, e_{m+n})$ so $\mathrm{Pic}(X) = \mathbb{Z} \times \mathbb{Z}$. Note $\mathrm{Pic}(X) \simeq \mathbb{Z} \times \mathbb{Z}$. Then

$$D_1 \dots D_{m+n} \quad = \quad \sum_{i_1, \dots, j_n \text{ distinct}} d_{i_1} \dots d_{i_m} e_{j_1} \dots e_{j_n}$$

Proof. We prove 1., 2. is similar. Note that $D_i \sim d_i H$ where $H_1 = V(x_1)$ a hyperplane, since divisor classes on $\mathbb{P}^n$ are classified by degree. So we have $D_1 \dots D_n = (d_1 H_1)\dots(d_n H_1) = d_1 \dots d_n (H_1^n) = d_1 \dots d_n (H_1 \dots H_n) = d_1 \dots d_n$, since $H_1 \dots H_n = 1$ by example 21.4. □

Example 21.6. To illustrate case 2., let $X = \mathbb{P}^1 \times \mathbb{P}^1$, π_1, π_2 be projections onto the first and second factor. Consider divisors on X, $D = \pi_1^* p$, $E = \pi_2^* q$ for some points $p, q \in \mathbb{P}^1$, then $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}D \oplus \mathbb{Z}E$, $D^2 = 1, E^2 = 1, D \cdot E = 1$, extending linearly

$$(d_1 D + e_1 E)\cdot(d_2 D + e_2 E) \quad = \quad e_1 d_2 D \cdot E + d_1 e_2 D \cdot E \quad = \quad d_1 e_2 + d_2 e_1$$

22. Applications of Bézout's theorem

Let $C \subseteq \mathbb{P}^2$ be a degree d curve $C = V(f)$, irreducible.

Definition 22.1. *A line $L \subseteq \mathbb{P}^2_{x,y,z}$ is an inflexion line of C if it has intersection multiplicity $\geqslant 3$ at some smooth point $p \in C$, that is if $\iota : L \longrightarrow \mathbb{P}^2$ then $\iota^* C = 3p +$effective divisor. The Hessian of f is the homogeneous polynomials*

$$H(x,y,z) \quad = \quad \begin{vmatrix} f_{xx} & f_{yx} & f_{zx} \\ f_{xy} & f_{yy} & f_{zy} \\ f_{xz} & f_{yz} & f_{zz} \end{vmatrix} \qquad f_{xy} := \frac{\partial^2 f}{\partial x \partial y}, \dots$$

The Hessian curve of f is C_H (zeroes of H). It depends on C only.

Theorem 22.2. *Suppose $C = V(f)$, where f is irreducible, is a smooth curve in $\mathbb{P}^2_{x,y,z}$ of degree $d \geqslant 3$. Then the inflexion points are the intersections poings of C with the Hessian C_H.*

Proof. Consider $p \in C$. Exercise: Hessian curve is independent of coordinatnes chosen, so we can change them such that $p = (1:0:0)$ and the tangent to C at p is $L = V(z)$. Write $f = f_0 + f_1 z + f_2 z$ where $f_0, f_1, f_2 \in \mathbb{K}[x,y]$. Recall that tangency mean $\iota^* C = 2p +$ effective ($\iota : L \longrightarrow \mathbb{P}^2$) and inflexion means $\iota^* C = 3p +$ effective. Therefore, on restricting to L, $f(x,y,0) = f_0$. Double root at p implies $f_0 = y^2 \widetilde{f}_0$, where $\widetilde{f}_0 \in \mathbb{K}[x,y]$. Inflexion point at p means $y^3 | f_0$, since $(1:0:0)$ is a smooth point, coefficient of $x^{d-1}z$ is nonzero, i.e. $f_{xz}(p) \neq 0$

$$H(1:0:0) \quad = \quad \begin{vmatrix} 0 & 0 & \# \\ 0 & f_{yy}(p) & * \\ \# & * & * \end{vmatrix} \qquad \text{where } \# \text{ non-zer}$$

Therefore $H(1:0:0)$ is zero iff $f_{yy}(p) = 0 \Longleftrightarrow$ constant term of $\tilde{f}_0 = 0$. That is $y^3 | f_0$. Therefore being an inflexion point is equivalent to being on the Hessian curve. □

Corollary 22.3. *A smooth plane curve $C \subset \mathbb{P}^2$ of degree d has at most $3d(d-2)$ inflexion points.*

Proof. The number of points is bounded above by Bézout. $C \cdot C_H = d(\deg(C_H) = 3d(d-2)$. □

22.1. **Common tangents to two conics.** Let C_1, C_2 be smooth conics (i.e. deg 2) in $\mathbb{P}^2_{x,y,z}$. What is the number of common tangents to C_1 and C_2. The answer we expect is 4.
Recall a line $ax + by + cz = 0$ in $\mathbb{P}^2_{x,y,z}$ corresponds to the point $(a,b,c) \in \mathbb{P}^{2\vee}_{a,b,c}$.
Describe the set of points in $\mathbb{P}^2_{a,b,c}$ correspondinng to tangents of C. For simplicity, change coordinates so that C_1 is $x^2 + y^2 + z^2 = 0$. Let's intersect with line L $ax + by + cz = 0$ with C_1. Consider affine open where $c \neq 0$. Intersection

$$\begin{aligned} 0 \quad &= \quad c^2\left(x^2 + y^2 + \left(\frac{-a}{c}x - \frac{b}{c}y\right)^2\right) \\ &= \quad c^2x^2 + c^2y^2 + a^2x^2 + 2abxy + b^2y^2 \\ &= \quad (a^2 + c^2)x^2 + 2abxy + (b^2 + c^2)y^2 \end{aligned}$$

Want a double root for C_1 and L to be tangent, i.e. discriminant, $\Delta = 0$

$$\Delta \quad = \quad 4a^2b^2 - 4(a^2 + c^2)(b^2 + c^2) \quad = \quad -4c^2(b^2 + a^2 + c^2) \quad = \quad 0$$

This is the condition for L to be tangent (on $c \neq 0$). Considering the affine pieces, $b \neq 0, c \neq 0$ we see te general condition is $a^2 + b^2 + c^2 = 0$. Therefore the set of tangent lines to C_1 corresponds to the conic $a^2 + b^2 + c^2 = 0$ in $\mathbb{P}^2_{a,b,c}$. The set of tangent lines to C_2 corresponds to another conic, intersecting these give the common tangents, which by Bézout's theorem is less than or equal to 4. In fact, in general it will be 4.
General situation: see exam problem list. What about the number of common tangents to two curves of arbitrary degree. Sketch: Define the dual curve $C^\vee := \{\ell \in \mathbb{P}^{2\vee}_{a,b,c} \mid \ell \text{ tangent to } C\}$. Can check that this is a subvariety of $\mathbb{P}^{2\vee}_{a,b,c}$, (find regular map $C \longrightarrow C^\vee$, $p \longmapsto$tangent line at p.) As before, common tangents to C_1 and C_2 is given by intersection points, $C_1^\vee \cap C_2^\vee$. Key is to determine the degree of $C^\vee$ and use

Bézout's theorem. To find the degree of $C_1^\vee$ (resp. $C_2^\vee$), we find the number of points of intersection of $C_1^\vee$ (resp. $C_2^\vee$) and any line in $\mathbb{P}^2_{a,b,c}$.
Exercise: a line, ℓ, in $\mathbb{P}^2_{a,b,c}$ corresponds to all lines in $\mathbb{P}^2_{x,y,z}$ though some fixed point (α, β, γ) (projective duality). To count these, recall that the tangent to $(x_0, y_0, z_0) \in C$ is

$$\frac{\partial f}{\partial x}(x_0 : y_0 : z_0)x + \frac{\partial f}{\partial y}(x_0 : y_0 : z_0)y + \frac{\partial f}{\partial z}(x_0 : y_0 : z_0)z \quad = \quad 0$$

Therefore a line passing through (α, β, γ) means that it is on the *polar curve*

$$C_p \quad : \quad \frac{\partial f}{\partial x}(x_0 : y_0 : z_0)\alpha + \frac{\partial f}{\partial y}(x_0 : y_0 : z_0)\beta + \frac{\partial f}{\partial z}(x_0 : y_0 : z_0)\gamma \quad = \quad 0$$

Therefore the number of these tangent lines through (α, β, γ) is in general $C \cdot C_p = d(d-1) = \deg(C^\vee)$.

23. More applications of intersection theory

23.1. **Pascal's Hexagon.**

Theorem 23.1. *Let $\overline{C} \subset \mathbb{P}^2$ be a plane conic (that is, irreducible of degree 2) $A, B, C, A', B', C' \in \overline{C}$ distinct. Let $P = AB' \cap A'B$, $Q = AC' \cap A'C$, and $R = BC' \cap B'C$. Then P, Q, R are collinear. (see geometry notes for picture)*

Proof. Let $D \in \mathrm{Div}(\mathbb{P}^2)$ sum of lines AB', BC', CA'. Similarly $E := A'B + B'C + C'A$. Note $\mathrm{supp}(D) \cap \mathrm{supp}(E) = \{A, A', B, B', C, C', P, Q, R\}$.
Let $D = V(f)$, $E = V(g)$ for 2 cubic forms f and g. Pick $O \in \overline{C} - \{A, A', B, B', C, C'\}$. Note then we have $f(O) \neq 0 \neq g(O)$. Therefore we can pick $\lambda \in \mathbb{K}$ such that

$$f(O) + \lambda g(O) \quad = \quad 0$$

Let $F = V(f + \lambda g)$, and we have

(1) $O \in F$,
(2) $\mathrm{supp}(D) \cap \mathrm{supp}(E) \subset F$ since for all $p \in \mathrm{supp}(D), \mathrm{supp}(E)$ $f(p) = 0 = g(p)$. Therefore $(f + \lambda g)(p) = 0$ etc.

If F and $\overline{C}$ are in general position, no. points in $\mathrm{supp}(F) \cap \mathrm{supp}(\overline{C}) \leqslant F \cdot \overline{C} \overset{\text{Bezout}}{=} 6$. But we have $\geqslant 7$ points $\{A, A', B, B', C, C', O\}$. Therefore F and $\overline{C}$ are not in general position, must have common components. Now $\overline{C}$ is irreducible, so there is only 1 component. Therefore $F = \overline{C} + $ line ℓ. Note that Bezout's theorem implies $P, Q, R \notin C$, but $P, Q, R \in \mathrm{supp}(F)$, therefore lies in line ℓ. □

23.2. **Blowing up varieties.** Let $p \in \mathbb{P}^n$, recall the blow-up at p $\mathrm{Bl}_p \mathbb{P}^n \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ can be though of as the "graph of porjection from p." Suppose $X \subseteq \mathbb{P}^n$ is a subvariety with $X \ni p$. Define the strict transform of X in $\mathrm{Bl}_p(\mathbb{P}^n)$ to be

$$s(X) \quad = \quad \overline{\sigma^{-1}(X - p)}$$

where $\sigma : \mathrm{Bl}_p \mathbb{P}^n \longrightarrow \mathbb{P}^n$.

Definition 23.2. *If p is a smooth point of X then we define the blowup of X at p to be*

$$\sigma : s(X) =: \mathrm{Bl}_p(X) \quad \longrightarrow \quad X$$

(1) The blowup $\sigma : \mathrm{Bl}_p X \longrightarrow X$ is an isomorphism away from p. Also $\mathrm{Bl}_p(X) \supset \sigma^{-1}(p) \simeq \mathbb{P}^{d-1}$ where $d = \dim(X)$.
(2) $\mathrm{Bl}_p(X)$ is smooth if X is smooth.
(3) $\mathrm{Bl}_p(X)$ is independent of the choice of embedding $X \subseteq \mathbb{P}^n$.
(4) If $Y \subseteq_{\text{closed}} X$ then $s(Y) \subseteq s(X)$, then $s(Y) \subseteq s(X)$ so we can speak of the strict transform of Y in $s(X) = \mathrm{Bl}_p(X)$.

23.3. **Picard group of blowups.** Let X be a smooth projective variety $p \in X$. Blowup

$$\begin{array}{rccc} \sigma : & \mathrm{Bl}_p X & \longrightarrow & X \\ & \cup & & \\ E := & \sigma^{-1}(p) & \longmapsto & p \end{array}$$

E is called an exceptional divisor. We have the map

$$\sigma^* : \mathrm{Pic}(X) \quad \longrightarrow \quad \mathrm{Pic}(\mathrm{Bl}_p(X))$$

Theorem 23.3. *In $\mathrm{Pic}(\mathrm{Bl}_p(X))$, so $\mathrm{Pic}(\mathrm{Bl}_p(X)) \simeq \mathbb{Z}E \oplus \sigma^* \mathrm{Pic}(X) \simeq \mathbb{Z}E \oplus \mathrm{Pic}(X)$, since σ^* is injective. (Proof omitted)*

Example 23.4. Let $X = \mathbb{P}^2$, $\mathrm{Pic}(X) = \mathbb{Z}$. We can compute $\sigma^* D = s(D) + E$.

23.4. Intersection theory on blown up surfaces.

Theorem 23.5. *Let X =smooth projective surface, $p \in X$, and $\sigma : \mathrm{Bl}_p(X) \longrightarrow X$ be the blowup at p. Let $E = \sigma^{-1}(p)$. Let $E = \sigma^{-1}(p)$ be the exceptional divisor (or curve). Then $E^2 = -1$.*

Reason: $1 = D \cdot D' = \sigma^*(D) \cdot \sigma^*(D')$, since we can move D, D' away from p where $\mathbb{P}^2$ and $\mathrm{Bl}_p \mathbb{P}^2$ are isomorphic. Therefore local intersection numbers are the same

$$\begin{aligned} 1 = D \cdot D' = \sigma^*(D) \cdot \sigma^*(D') &= (s(D) + E) \cdot (s(D') + E) \\ &= s(D) \cdot s(D') + E \cdot s(D') + s(D) \cdot E + E^2 \\ &= 0 + 1 + 1 + E^2 \\ E^2 &= -1 \end{aligned}$$

Corollary 23.6. *We know*

$$\mathrm{Pic}\left(\mathbb{P}^1 \times \mathbb{P}^1\right) \simeq \mathrm{Pic}\left(\mathrm{Bl}_p \mathbb{P}^2\right) \simeq \mathbb{Z} \times \mathbb{Z}$$

but the intersection pairing

$$\cap : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

is different so $\mathbb{P}^1 \times \mathbb{P}^1 \not\simeq \mathrm{Bl}_p \mathbb{P}^2$. More precisely, $\mathbb{P}^1 \times \mathbb{P}^1$ has no effective curve E with $E^2 = E \cdot E = -1$. Let $\mathrm{Pic}\left(\mathbb{P}^1 \times \mathbb{P}^1\right) = \mathbb{Z}D \oplus \mathbb{Z}E$ where D and E are some generators. An effective curve has the form $aD + bE$ for some $a, b \in \mathbb{N}$, and it is clear that $(aD + bE)^2 \geqslant 0$.

24. Birational Classification of Projective Varieties

24.1. Program to understand all projective varieties.

(1) *Classification*: obtain a list of all projective varieties up to some equivalence relation.
(2) *Criteria*: given projective variety X, find effective criteria to check which variety on your list X is.
(3) *Properties*: determine all geometric properties of varieties on your list, e.g. Picard group.

We usually do (1), (2), (3) together. This program is reasonably complete in

(1) Dimension 1 (projective curves) Classical 19th century stuff done by the German school.
(2) Dimension 2 (surfaces) Early 20th century, Italian school (Enriques, Castelnuovo).
(3) Dimension 3 (3-folds) 1980's by Japanese school and others (Mori, Reid, ...)

Still lots of research in dimension $\geqslant 4$.

24.2. Birational Equivalence. The best case scenario is to classify varieties up to isomorphism. But usually it is easier to classify up to birational equivalence, i.e. identify birational varieties. Recall that given a dominant rational map of varieties $\varphi : X - \rightarrow Y$ we get a field homomorphism $\varphi^* : \mathbb{K}(Y) \longrightarrow \mathbb{K}(X)$ of field of rational functions. Converse is true too:

Proposition 24.1. *Let X, Y be quasi-projective varieties. Let $\psi : \mathbb{K}(Y) \longrightarrow \mathbb{K}(X)$ be a field homomorphism. Then there is a unique dominant rational map $\varphi : X - \rightarrow Y$ such that $\varphi^* = \psi$.*

Proof. Since φ only rational, we can assume X, Y affine. Suppose $\mathbb{K}[Y]$ is finitely generated as a $\mathbb{K}$-algebra by $y_1, \cdots, y_n$. By clearing denominators, we can find $x \in \mathbb{K}[X]$ such that $x\psi(y_i) \in \mathbb{K}[X]$ for all i. Let U be an open subset of X where $x \neq 0$. Then ψ restricts to a map $\psi|_{\mathbb{K}[Y]} : \mathbb{K}[Y] \longrightarrow (\mathbb{K}[X])[X^{-1}] \simeq \mathbb{K}[U]$. Hence obtain a regular map $U \longrightarrow Y$, therefore rational map $\varphi : X - \rightarrow Y$ such that $\varphi^* = \psi$ (Ex: check φ dominant). □

Corollary 24.2. *Let $\varphi : X - \rightarrow Y$ be a dominant rational map. The following are equivalent:*

(1) φ is birational
(2) φ restricts to an isomorphism of an open subsets of X with an open subset of Y
(3) $\varphi^* : \mathbb{K}(Y) \longrightarrow \mathbb{K}(X)$ is a field isomorphism.

24.3. Reduction of classification problem to smooth projective varieties. A quasi-projective variety is birational with its closure in projective space, so we'll only classify projective varieties X. Smooth varieties are easier to study, since like manifolds, so can apply techniques of differential geometry.

Definition 24.3. *Let X =projective variety. A* resolution of singularities *(for X) is a smooth projective variety $\tilde{X}$ and a birational morphism (i.e. regular map) $\pi : \tilde{X} \longrightarrow X$. If π is not birational but is still dominant rational, and $\pi^* : \mathbb{K}(Y) \longrightarrow \mathbb{K}(X)$ is a finite field extension, then say π is an alteration.*

Example 24.4. $X \subseteq \mathbb{P}^2_{x,y,z}$, $X = V(zy^2 - x^3)$. We'll show $\pi : \mathbb{P}^1_{u,v} \longrightarrow X$, $(u : v) \longmapsto (u^2 v : u^3 : v^3)$ is a resolution of singularities. (Note $\mathbb{P}^1$ smooth) Its rational inverse is $\varphi : X - \rightarrow \mathbb{P}^1_{u,v}$, $(x : y : z) \longmapsto (y : x)$ which is defined when y or $x \neq 0$.

24.4. **Resolution of Singularities.** Let X =projective variety.

Theorem 24.5. *X has a resolution of singularities if*

(1) $\dim(X) = 1$ *(classical 19th century)*
(2) $\dim(X) = 2$ *(due to Walker 1935, Zariski 1939,* $\mathrm{char}(\mathbb{K}) = 0$*. Abhyankar 1956* $\mathrm{char}(\mathbb{K}) > 0$*)*
(3) $\mathrm{char}(\mathbb{K}) = 0$ *(Hironaka 1964)*
(4) For arbitrary $\mathrm{char}(\mathbb{K})$*, X has an alteration (de Jong 1995).*

Upshot:

1. If only interested in birational classification then for dim $\leqslant 2$ or char $= 0$, suffice to classify smooth varieties and determine which are birational.
2. If want classification up to isomorphism, then better study resolution of singularities $\hat{X} \longrightarrow X$ too.

Remark: conjecturally all projective varieties have resolutions of singularities.
Recall corollary of lecture 18, that birational smooth projective curves are isomorphic. This implies that projective curvers have unique resolution of singularities, and the classification of {projective curves}/birational equivalence is same as classification of {smooth projective curves}/isomorphism. This is completely false in higher dimensions. If X is a smooth projective variety of dimension $d \geqslant 2$ then so is $\mathrm{Bl}_p X$ which is birational to X but not isomorphic, since exceptional divisor is a fibre $\simeq \mathbb{P}^{d-1} \neq$point.

25. Differential Forms

Let X =smooth projective variety. Consider U =affine variety. Recall given $p \in U$, there is surjective linear map $\mathrm{d}_p : \mathbb{K}[U] \longrightarrow T^*_{U,p}$, $f \longmapsto \mathrm{d}_p f$. Now think of f fixed and vary p to get function which assigns to $p \in U$, $\mathrm{d}_p f \in T^*_{U,p}$ ("covector field").

25.1. **Regular Differential 1-Forms.**

Definition 25.1. *A* **regular differential** 1**-form** *on X is a function which assigns to $p \in X$, $\omega_p \in T^*_{X,p}$ such that Zariski-locally it has form $\sum g_i \mathrm{d} f_i$, g_i, f_i regular functions, i.e. for any $p_0 \in X$ there is some affine open neighbourhood U and $f_i, g_i \in \mathbb{K}[U]$ such that for any $p \in U$, $\omega_p = \sum g_i(p)\mathrm{d}_p f_i$. Let $\Omega^1[X]$ denote set of such 1-forms.*

Example 25.2. $X = V(Y^2Z - X(X-Z)(X+Z)) \subseteq \mathbb{P}^2_{X,Y,Z}$. Look at affine piece where $Z \neq 0$, let $X = \frac{X}{Z}$, $Y = \frac{Y}{Z}$. Here X is subvariety of $\mathbb{A}^2_{x,y}$ given by $y^2 = x(x-1)(x+1) = x^3 - x$. Therefore $2y\mathrm{d}y = (3x^2-1)\mathrm{d}x$. Consider differential form $\omega = \frac{\mathrm{d}x}{y} = \frac{2\mathrm{d}y}{3x^2-1}$. $\frac{\mathrm{d}x}{y}$ is regular whenever $y \neq 0$, i.e. $x^3 - x \neq 0$. $\frac{2\mathrm{d}y}{3x^2-1}$ is regular wherever $3x^2 - 1 \neq 0$, i.e. $x \neq \pm\sqrt{1/3}$. Therefore ω is certainly regular if $Z \neq 0$. Therefore only need to check regularity at $(0:1:0)$. Ex: check similarly that ω is also regular at $(0:1:0)$. Hint: consider affine piece $Y \neq 0$ and use coordinates $u = \frac{X}{Y}$, $v = \frac{Z}{Y}$ and rewrite $\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}(u/v)}{v^{-1}} = \cdots =$reg $\mathrm{d}u$+reg $\mathrm{d}v$.

Remark 25.3. If U =affine variety, $\omega, \omega' \in \Omega^1[U], f \in \mathbb{K}[U]$, then $\omega + \omega' \in \Omega^1[U]$ and $fw \in \Omega^1[U]$. In fact, $\Omega^1[U]$ is a $\mathbb{K}[U]$-module. For X not affine, we still have $\Omega^1[X]$ is a vector space over $\mathbb{K}$.

Theorem 25.4. *Let X =smooth quasi-projective variety, dim n, $p \in X$ and $f_1, \cdots, f_n \in \mathbb{K}(X)$ regular at p. Suppose that $\{\mathrm{d}_p f_1, \cdots, \mathrm{d}_p f_n\} \subset T^*_{X,p}$ is a basis (/$\mathbb{K}$). Then there is an affine open neighbourhood $U \ni p$ such that $f_i \in \mathbb{K}[U]$, $\Omega^1[U] = \mathbb{K}[U]\mathrm{d}f_1 \oplus \cdots \oplus \mathbb{K}[U]\mathrm{d}f_n$.*

25.2. **Rational Differential 1-Forms.**

Definition 25.5. *A* **rational differential** 1**-form** *on X is a function ω, defined on some dense open $U \subset X$ mapping $p \in U$ to $\omega_p \in T^*_{X,p}$ such that locally it has the form $\sum_{i=1}^l g_i \mathrm{d} f_i$ with f_i regular and $g_i \in \mathbb{K}(X)$. Denote the set of such 1-forms by $\Omega^1(X)$. As for rational functions, we identify rational 1-forms if they coincide on a dense open set.*

Corollary 25.6. *If $U \subset X$, U open and dense in X, then $\Omega^1(U) = \Omega^1(X)$.*

Thm $\Longrightarrow \Omega^1(X)$ is a vector space over $\mathbb{K}(X)$ of $\dim n = \dim X$. In notation of theorem, $\Omega^1(X) = \Omega^1(U) = \mathbb{K}(X)\mathrm{d}f_1 \oplus \cdots \oplus \mathbb{K}(X)\mathrm{d}f_n$.

Example 25.7. $X = \mathbb{P}^1_{x,y}$. Let $z = \frac{y}{x}$. z is regular away from $p = (0:1)$. Can check $\mathrm{d}z$ is a basis for $T_{\mathbb{P}^1,s}$ for $s \in \mathbb{P}^1 - p$. Fact: If $U = \mathbb{P}^1 - p$, then $\mathbb{K}[U] = \mathbb{K}[z]$ and $\Omega^1[U] = \mathbb{K}[z]\mathrm{d}z$. So $\Omega^1(\mathbb{P}^1) = \Omega^1(U) = \mathbb{K}(z)\mathrm{d}z$. Fact: $\Omega^1[\mathbb{P}^1] = 0$. Why? Suppose $\omega = g(z)\mathrm{d}z \in \Omega^1[\mathbb{P}^1]$, $g \in \mathbb{K}[z]$. Now check regularity at p. Introduce $t = z^{-1}$ and note $\mathrm{d}_p t \neq 0$ so Ω^1 is generated by $\mathrm{d}t$ in neighbourhood of p by theorem. $w = g(z)\mathrm{d}z = (a_n t^{-n} + \cdots + a_0)(-t^{-2}\mathrm{d}t)$ in any neighbourhood of p. This is not a regular function unless is 0. Therefore $\omega = 0$ and $\Omega^1[\mathbb{P}^1] = 0$.

25.3. **Functoriality.** Given regular map $\varphi : X \longrightarrow Y$, get induced linear map $\varphi^* : \Omega^1[Y] \longrightarrow \Omega^1[X]$ defined essentially by $\varphi^*(\sum g_i \mathrm{d}f_i) = \sum \varphi^*(g_i)\mathrm{d}(\varphi^* f_i)$.

Theorem 25.8.

(1) If X =smooth projective variety, then $\dim_{\mathbb{K}} \Omega^1[X] < \infty$.
(2) Given a dominant rational map $\varphi : X - \to Y$ (say from regular map $U \longrightarrow Y$, U open in X) we get $\varphi^* : \Omega^1[Y] \longrightarrow \Omega^1[U]$
*(3) In particular, we see that $\Omega^1[X]$ is a birational invariant. We call $\dim_{\mathbb{K}} \Omega^1[X] = g(X)$ the **genus** of X.*

26. Canonical divisor

Let X = smooth projective curve, ω =rational differential 1-form. Aim: To associate to ω a divisors D_ω which changes only up to linear equivalence as you change ω.3
As usual, to define D_ω need only define on an affien open cover $\{U_\alpha\}$ of X. Let $p \in X$, last lecture saw that there is an affine neighbourhood U containing p such that $\Omega^1[U] = \mathbb{K}[U]\mathrm{d}t$ where $t \in \mathbb{K}[U]$. Also $\Omega^1(X) = \Omega^1(U) = \mathbb{K}(U)\mathrm{d}t \ni \omega = g\mathrm{d}t$ for some $g \in \mathbb{K}(U)$. We will define D_ω only U to be $\mathrm{div}(g)$. This is independent of the cohice of t in the following sense,

Proposition 26.1. *Suppose also $s \in \mathbb{K}[U]$ with $\Omega^1[U] = \mathbb{K}[U]\mathrm{d}s$. If $\omega = h\mathrm{d}s$, then $\mathrm{div}(h) = \mathrm{div}(g)$.*

Proof. Note $\mathbb{K}[U]\mathrm{d}t = \mathbb{K}[U]\mathrm{d}s$, so $\mathrm{d}t = u\mathrm{d}s$ and $u \in \mathbb{K}[U]$ is invertible, sometimes we write $u =: \frac{\mathrm{d}t}{\mathrm{d}s}$. $g\mathrm{d}t = gu\mathrm{d}s = h\mathrm{d}s$. □

Note that u invertible implies $\mathrm{div}(u) = 0$ since u and u^{-1} have no poles. Therefore $\mathrm{div}(g) = \mathrm{div}(g) + \mathrm{div}(u)$. This gives a well defined D_ω.

Example 26.2. Consider $\mathbb{P}^1_{x,y}$ and $z = y/x$ is regular where $x \neq 0$, $\omega = \mathrm{d}z$. What is D_ω? Look on affine pieces $U_0 := \{x \neq 0\} \simeq \mathbb{A}^1_z$, $\Omega'[U] = \mathbb{K}[U]\mathrm{d}z = \mathbb{K}[z]\mathrm{d}z$. Therefore $D_\omega|_U = \mathrm{div}(1) = 0$. On $\{y \neq 0\}$, $t = z^{-1}$ is regular and $\Omega^1[V] = \mathbb{K}[V]\mathrm{d}t$, $\omega = \mathrm{d}(t^{-1}) = -t^{-2}\mathrm{d}t$, therefore $D_\omega|_V = \mathrm{div}(-t^{-2}) = -2p$ where $p = (0:1)$. Therefore on $\mathbb{P}^1$, $D_\omega = -2p$.

26.1. **Canonical divisor.** A canonical divisor on X is the divisor class containing D_ω. This is well defined- we show that chaning ω to $\eta \in \Omega^1(X)$ implies $D_\omega \sim D_\eta$. Note that $\Omega^1(X) = \mathbb{K}(X)\mathrm{d}t$ for some $t \in \mathbb{K}(X)$, hence $\eta = f\omega$ for some $f \in \mathbb{K}(X)$. By definition

$$D_\eta = D_\omega + \mathrm{div}(f)$$

Example 26.3. Continue from above. Let $X = \mathbb{P}^1$. Identifying $\mathrm{Pic}(\mathbb{P}^1) \xrightarrow{\deg} \mathbb{Z}$ we see $\deg(K_{\mathbb{P}^1}) = -2$.

26.2. **Riemann-Roch space.** Let Y be a quasi-projective variety $D \in \mathrm{Div}(Y)$ define the Riemann-Roch space associated to D to be

$$\mathcal{L}(D) = \{f \in \mathbb{K}(Y)^* \mid \mathrm{div}(f) + D \geqslant 0\} \cup \{0\}$$

This is a vector space over $\mathbb{K}$, so denote $\ell(D) := \dim_{\mathbb{K}}(\mathcal{L}(D))$.

Example 26.4. $X = \mathbb{P}^1_{x,y}$, $z = x/y$, let $D = p$. Note that $z \in \mathcal{L}(D)$ since $\mathrm{div}(z) + D = (q-p) + p = q \geqslant 0$. Also constants are in $\mathcal{L}(D)$. In fact, $\mathcal{L}(D) = \mathbb{K}1 \oplus \mathbb{K}z$. Why? If $f(z) = g(z)/h(z) \in \mathcal{L}(D)$ with $g, h \in \mathbb{K}[z]$ relatively prime then must have h =constant and $\deg(g) \leqslant 1$. (else have higher order pole at $x = 0$).

Proposition 26.5.

(1) If $D_1 \sim D_2$ say $D_1 = D_2 + \mathrm{div}(g)$ for $g \in \mathbb{K}(Y)^*$ then

$$g\mathcal{L}(D_1) = \mathcal{L}(D_2)$$

and $\ell(D_1) = \ell(D_2)$.
(2) $g(X) = \ell(\mathbb{K}_X)$ (g is the genus of X) when X is a smooth projective curve since

$$\mathcal{L}(D_\omega)\omega = \Omega^1[X]$$

Proof. (1) $\mathcal{L}(D_1) = \{f \mid \mathrm{div}(f) + D_1 \geqslant 0\}$, $\mathrm{div}(f) + D_2 + \mathrm{div}(g) = \mathrm{div}(fg) + D_2$. □

26.3. **The Riemann-Roch theorem.** Let X be a smooth projective curve, then for $D \in \mathrm{Div}(X)$,

$$\ell(D) - \ell(K_X - D) \quad = \quad \deg(D) - g(X) + 1$$

Proof. Non-trivial. □

We can use the Riemann-Roch theorem to classify smooth projective curves. We have a discrete invariant $g(X) \in \mathbb{N}$.

Corollary 26.6. *Let X be a smooth projective curve $g(X) = 0$ iff $X \simeq \mathbb{P}^1$.*

Proof. We have shown $g(\mathbb{P}^1) = 0$. Applying Riemann-Roch to the divisor p gives

$$\begin{aligned} \ell(p) - \ell(K_X - p) &= 1 - g(X) + 1 \\ \ell(p) &= 2 \end{aligned}$$

so there exists $f \in \mathcal{L}(p)$ nonconstant and $f : X \longrightarrow \mathbb{P}^1$ is a regular map with a simple pole, i.e. f is an isomorphism (needs further proof). □

However there are lots of non-isomorphic smooth projective curves of genus 1.

27. Canonical divisor in higher dimensions

Let X be a smooth projective variety, with dimension d. Last time, we saw that if $\dim(X) = 1$, then X is covered by affine opens U such that $\Omega^1[U] = \mathbb{K}[U]\mathrm{d}t$ for some $t \in \mathbb{K}[U]$ and $\Omega^1(U) = \Omega^1(X) = \mathbb{K}(U)\Omega^1[U] = \mathbb{K}(U)\mathrm{d}t$.
If $0 \neq \omega \in \Omega^1(X)$, then we can use $\Omega^1(U) \sim \mathbb{K}(U)$ (the isomorphism changes with U) to determine D_ω.
Question: How to extend this to higher dimensional varieties where now $\Omega^1[U]\mathrm{d}t_1 \oplus \ldots \oplus \mathbb{K}[U]\mathrm{d}t_d$. Answer: Use

27.1. **Exterior products.** Let R be a commutative ring, and$F$ be the free R-module with basis $e_1, \ldots, e_d$, that is $F = Re_1 \oplus \ldots \oplus Re_d$. Exterior product defined to be the algebra

$$\bigwedge^* F \quad = \quad \frac{R\langle e_1, \ldots, e_d\rangle}{\langle e_ie_j + e_je_i\rangle}$$

that is, in $\bigwedge^* F$, we have $e_ie_j = -e_je_i$ and $e_i^2 = 0$. We write for $v_1, \ldots, v_r \in F$, $v_1 \wedge v_2 \wedge \ldots \wedge v_r$ for the image of $v_1 \ldots v_r$ in $\bigwedge^* F$.
Facts: there is the R-module isomorphism

$$\bigwedge^* F \quad \simeq \quad \bigoplus \bigwedge^r F$$

where $\bigwedge^r F = \bigoplus_{i_1 < \ldots < i_r} Re_{i_1} \wedge \ldots \wedge Re_{i_r}$ is an R-submodule of $\bigwedge^r F$.

27.2. **Universal property.**

(1) The map

$$\begin{aligned} F^r &\longrightarrow \bigwedge^r F \\ (v_1, \ldots, v_r) &\longmapsto v_1 \wedge \ldots \wedge v_r \end{aligned}$$

is R-linear in each variable v_i, and is alternating.

(2) $\bigwedge^r F$ (and hence $\bigwedge^* F$) is independent of choice of coordinates since it is universal with property 1.

27.3. **Functoriality.** Let $\varphi : F \longrightarrow F'$ be an homomorphism of finitely generated free R-modules. Then we obtain an R-module homomorphism

$$\begin{aligned} \bigwedge^r \varphi : \bigwedge^r F &\longrightarrow \bigwedge^r F' \\ e_{i_1} \wedge \ldots \wedge e_{i_r} &\longmapsto \varphi(e_{i_1}) \wedge \ldots \wedge \varphi(e_{i_r}) \end{aligned}$$

independent of choice of coordaintes as $v_1 \wedge \ldots \wedge v_r \longmapsto \varphi(v_1) \wedge \ldots \wedge \varphi(v_r)$

27.4. **Canonical divisor.**

Definition 27.1. *Elements in here are called rational differential r-forms*$\rightarrow \Omega^r(X) = \bigwedge^r \Omega^1(X) \leftarrow \mathbb{K}(X)$*-vector space (using* $R = \mathbb{K}(X)$*).*

Choose $U \subset X$ an affine open subset so that

$$\Omega^1[U] = \mathbb{K}[U]\mathrm{d}t_1 \oplus \ldots \oplus \mathbb{K}[U]\mathrm{d}t_d$$

where $d = \dim(X)$. Define

$$\Omega^r[U] = \bigwedge^r \Omega^1[U]$$

where $R = \mathbb{K}[U]$, to be the regular differential r-forms.
The above imply that

$$\begin{aligned}\Omega^d(X) &= \Omega^d(U)\\ &= \bigwedge^r (\mathbb{K}(U)\mathrm{d}t_1 \oplus \ldots \oplus \mathbb{K}(U)\mathrm{d}t_d)\\ &= \mathbb{K}(U)\mathrm{d}t_1 \wedge \ldots \wedge \mathrm{d}t_d\\ &\simeq \mathbb{K}(X)\mathrm{d}t_1 \wedge \ldots \wedge \mathrm{d}t_d\end{aligned}$$

and

$$\Omega^d[U] = \mathbb{K}[U]\mathrm{d}t_1 \wedge \ldots \wedge \mathrm{d}t_d$$

Fix $0 \neq \omega \in \Omega^d(X)$. To define $D_\omega \in \mathrm{Div}(X)$, define on U as follows: can write

$$\omega = f_U \mathrm{d}t_1 \wedge \ldots \wedge \mathrm{d}t_d$$

where $f_U \in D_\omega$ and D_ω on U is $\mathrm{div}(f)$ in U. A more complication version of argument from last lecture implies that changing basis $\mathrm{d}t_1, \ldots, \mathrm{d}t_d$ does not change $\mathrm{div}(f)$ on U. Therefore D_ω is well-defined. Also $[D_\omega] \in \mathrm{Pic}(X)$ is independent of ω and is called the **canonical divisor**, denoted K_X.

Example 27.2. Let $X = \mathbb{P}^2_{x,y,z}$. On $U = \{z \neq 0\}$, $u = x/z$, $v = y/z$ are regular on U, and $\mathbb{K}[U] = \mathbb{K}[u, v]$. Also, it turns out $\Omega^1[U] = \mathbb{K}[U]\mathrm{d}u \oplus \mathbb{K}[U]\mathrm{d}v$. Let $0 \neq \omega = \mathrm{d}u \wedge \mathrm{d}v \in \Omega^1(\mathbb{P}^2)$, D_ω on U is $\mathrm{div}(1) = 0$. On $V = \{y \neq 0\}$, $s = x/y, y = z/y$, so $\mathbb{K}[U] = \mathbb{K}[s, t]$,

$$\Omega^1[U] = \mathbb{K}[U]\mathrm{d}s \oplus \mathbb{K}[U]\mathrm{d}t$$

$\omega \in \mathbb{K}(U)\mathrm{d}s \wedge \mathrm{d}t$. Now

$$\begin{aligned}\omega &= \mathrm{d}u \wedge \mathrm{d}v\\ &= \mathrm{d}\left(\frac{s}{t}\right) \wedge \mathrm{d}\left(\frac{1}{t}\right)\\ &= \left(\frac{1}{t}\mathrm{d}s - \frac{s}{t^2}\mathrm{d}t\right) \wedge \left(-\frac{1}{t^2}\right)\mathrm{d}t\\ &= -\frac{1}{t^3}\mathrm{d}s \wedge \mathrm{d}t\end{aligned}$$

Theorem 27.3.
(1) *If we identify* $\mathrm{Pic}(\mathbb{P}^n)$ *with* $\mathbb{Z}$ *via degree, then* $\mathbb{K}_{\mathbb{P}^n} = -(n+1)$
(2) *Adjunction formula. Let* Y *be a smooth codimension 1 subvariety of* X *and* $\iota : Y \longrightarrow X$ *the inclusion. Then*

$$\begin{aligned}K_Y &= \iota^*(K_X + Y)\\ &=: K_X + Y|_Y\end{aligned}$$

(3) *Let* $Y \subset \mathbb{P}^2$ *be a smooth curve, then* $\deg(K_Y) = e(e-3)$*, where* e *is the degree of* Y*.*

Proof. Only do $2 \Longrightarrow 3$.

$$\begin{aligned}K_Y &= K_{\mathbb{P}^2} + Y|_Y\\ \deg(K_Y) &= (K_{\mathbb{P}^2} + Y) \cdot Y\\ &= (-3 + e)e\end{aligned}$$

□

Example 27.4.
(1) If $e = 1$, then $Y \simeq \mathbb{P}^1$, $\deg(K_Y) = -2$, know $K_{\mathbb{P}^1} = -2$.
(2) If $e = 2$, then $Y \simeq \mathbb{P}^1$ still.
(3) If $e = 3$, $\deg(K_Y) = 0$, and Y is not isomorphic to $\mathbb{P}^1$.

In fact, we see that smooth plane curves of different degree are non-isomorphic, since $\deg(K_Y)$ differs for these.

28. Pluricanonical maps

Let X be a smooth projective variety, and $D \in \mathrm{Div}(X)$. Consider the Riemann-Roch space

$$\mathcal{L}(D) \quad = \quad \{f \in \mathbb{K}(X)^* \mid \mathrm{div}(f) + D \geqslant 0\} \cup \{0\}$$

and let $f_0, f_1, \ldots, f_n$ be a basis over $\mathbb{K}$. Then we obtain a rational map associated to D (most of the time)

$$\begin{aligned} \varphi : X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto (f_0(x) : \ldots : f_n(x)) \end{aligned}$$

Suppose we change the basis $f_0, \ldots, f_n$ b $M \in GL_{n+1}(\mathbb{K})$, then φ only changes by a composite

$$X - \xrightarrow{\varphi} \mathbb{P}^n \xrightarrow{M} \mathbb{P}^n$$

Essentially, this map depends only on the divisor class of D.

Proposition 28.1.

(1) *Let $\varphi_m : X - \to \mathbb{P}^{n(m)}$ be the rational map associated to mK_X. This is called the m-**th pluri-canonical map**. The φ_m factors through φ_{m,m_2} in sense*

$$\begin{array}{ccc} X & - \to & \mathrm{im}(\varphi_{m_1}) \\ & \searrow & \mathrm{im}(\varphi_{m_1 m_2}) \\ & \searrow & \mathrm{im}(\varphi_{m_1 m_2 m_3)} \end{array}$$

(2) *Eventually these map stablise, i.e. for large m, we get "fixed map" $X - \to \mathrm{im}(\varphi_m) =: X_{\mathrm{can}}$. This map is called the **canonical map**. The **Kodaira dimension** of X is $\dim(X_{\mathrm{can}})$ denoted $\mathrm{Kod}(X)$.*

(3) *The Riemann-Roch spaces $\mathcal{L}(mK_X)$ are birational invariants so $\mathrm{Kod}(X)$ is too.*

28.1. Classification of varieties via Kodaira dimension.

28.1.1. *Case for curves.*

- $g = 0$. Then $X = \mathbb{P}^1$ here $\deg(K_{\mathbb{P}^1}) = 2$ therefore $\deg(mK_{\mathbb{P}^1}) = -2m$ where $m \geqslant 1$. Hence

$$\begin{aligned} \mathcal{L}(mK_{\mathbb{P}^1}) &= \{f \in \mathbb{K}(X)^* \mid \mathrm{div}(f) - 2m \geqslant 0\} \\ &= 0 \end{aligned}$$

  since $\mathrm{div}(f) = 0$. Pluricanonical maps do not exist, so define the Kodaira dimension to be $\mathrm{Kod}(\mathbb{P}^1) = \dim(\emptyset) = -\infty$.
- $g = 1$. That is $\ell(K_X) = 1$. In fact, $\mathcal{L}(mK_X) = 1$ and $K_X = 0$. Pluricanonical maps $X \longrightarrow \mathbb{P}^0 =: X_{\mathrm{can}} = \mathrm{pt}$. So the $\mathrm{Kod}(X) = 0$.
- $g \geqslant 2$. $\ell(K_X) \geqslant 2$, therefore have non proportional $f_0, f_1 \in \mathcal{L}(K_X)$ and the canonical map

$$\varphi_1 : X \quad \longrightarrow \quad \mathbb{P}^n$$

  does not map to a point. That is is nonconstant, so the image is one dimensional, and the image of pluricanonical maps also dimension 1. The $\mathrm{Kod}(X) = 1$, turns out that $X \longrightarrow X_{\mathrm{can}}$ is an isomorphism.

28.1.2. *Case for surfaces.* Let X be a smooth projective surface.

- $\mathrm{Kod}(X) = -\infty$. X is birational to $C \times \mathbb{P}^1$ where C is a curve. These are called ruled surfacs.

  Example 28.2. $\mathbb{P}^2$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ (Both contain $\mathbb{A}^2$ as an affine open), and $\mathrm{Kod}(\mathbb{P}^2) = -\infty$. These surfaces are very special but occur frequently in mathematics.

- $\mathrm{Kod}(X) = 0$. These have been classified into four types. For example, let X be a quartic in $\mathbb{P}^3$.

$$\begin{aligned} K_X &\overset{\text{adjunction}}{=} (K_{\mathbb{P}^3} + X)|_X \\ &= (-4 + 4)|_X \\ &= 0 \end{aligned}$$

  therefore $mK_X = 0$.

$$\begin{aligned} \mathcal{L}(mK) &= \mathcal{L}(0) \\ &= \{f \in \mathbb{K}(X)^* \mid \mathrm{div}(f) + 0 \geqslant 0\} \cup \{0\} \\ &= \text{regular functions on } X \\ &= \mathbb{K} \end{aligned}$$

  so X is projective. The pluricanonial map $X - \to \mathbb{P}^0$.
- $\mathrm{Kod}(X) = 1$. We have a rational map $X - \to C = X_{\mathrm{can}}$ where C is a curve. This fibration allows you to study X. It turns out the fibres are generically curves of genus 1, hence they are called elliptic fibrations.
- $\mathrm{Kod}(X) = 2$. These are the general surfaces, but rarely occur in mathematics. Still being classified. We can use canonical map $X - \to X_{\mathrm{can}}$ to study X.

When are two smooth projective surfaces birational? Ans:

Theorem 28.3. *(Zariski) Let $\varphi : X - \to Y$ be a birational map, then there is a diagram*

$$
\begin{array}{ccccccccccccc}
 & & & & & X_m & = & Y_m & & & & & \\
 & & & & \swarrow & & & & \searrow & & & & \\
 & & & X_{m-1} & & & & & & Y_{m-1} & & & \\
 & & \swarrow & & & & & & & & \searrow & & \\
 & X_{m-2} & & & & & & & & & & Y_{m-2} & \\
 \ddots & & & & & & & & & & & & \ddots \\
Y = X_1 & & & & & & - \to & & & & & & Y_1 = Y
\end{array}
$$

and $X_i \longrightarrow X_{i-1}$ and $Y_i \longrightarrow Y_{i-1}$ are blowups at points.

$$
\begin{array}{ccccccc}
 & & \mathrm{Bl}_p(\mathbb{P}^1 \times \mathbb{P}^1) & \simeq & \mathrm{Bl}_q(\mathrm{Bl}_q \mathbb{P}^2) & & \\
 & \swarrow & & & & \searrow & \\
\mathbb{P}^1 \times \mathbb{P}^1 & & - \to & \mathbb{P}^2 & \longleftarrow & & \mathrm{Bl}_p(\mathbb{P}^2)
\end{array}
$$