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MATH5765: Algebraic Geometry, Assignment 2

This assignment is also relatively straightforward and you should aim to get full marks for it. I encourage you to discuss the questions with classmates and I certainly hope you check your answers against each other. The definition of cheating for this course means writing something you don't understand.

As always, k is an algebraically closed field. The questions concentrate on materials in lectures 13-16 so brush up on those first.

1. Consider the subalgebra $S := k[x^3, x^2y, xy^2, y^3] \subset k[x, y]$.

   i. Show that the inclusions $k[x^3, y^3] \hookrightarrow S$ and $S \hookrightarrow k[x, y]$ are finite.

   ii. As in assignment 1, S corresponds to some affine variety say X. Using the previous part or otherwise, find the dimension of X.

2. Here are some basic facts about low dimensional varieties.

   i. Show that the only dimension zero quasi-projective varieties are points.

   ii. Describe all closed subsets of curves.

3. This question is concerned with an algebraic version of the Whitney embedding theorem. It is an elegant but typical application of dimension theory. Consider a projective curve C in $\mathbb{P}^n$. We wish to show that there is in fact a regular injective morphism $C \hookrightarrow \mathbb{P}^3$.

   Assume that C is not a line as otherwise it embeds in $\mathbb{P}^1$. Let $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal and $\Gamma = \text{Grass}(2, n+1)$ which classifies lines in $\mathbb{P}^n$.

   i. Let $\phi : \mathbb{P}^n \times \mathbb{P}^n - \Delta \longrightarrow \Gamma$ be the map which sends (x, y) to the line joining x and y. Show that ϕ is regular.

   ii. Restrict ϕ to $C \times C$ to obtain a regular map $\psi : C \times C - \Delta \longrightarrow \Gamma$. Find the dimension of the non-empty fibres of ψ.

iii. Let X be the closure (in Γ) of im ψ. Determine (with reason) the dimension of X.

iv. We consider the so-called "partial flag variety"

$$F := \{(x, l) \in \mathbb{P}^n \times \Gamma | x \in l\} \subset \mathbb{P}^n \times \Gamma.$$

Show that F is a closed subset of $\mathbb{P}^n \times \Gamma$.

v. Let $\pi_1 : F \longrightarrow \mathbb{P}^n$ and $\pi_2 : F \longrightarrow \Gamma$ be the projection maps. Show that $\pi_2^{-1}(X)$ is a projective variety and determine its dimension.

vi. Let $Y = \pi_1 \pi_2^{-1}(X) \subseteq \mathbb{P}^n$. Show it is a projective variety and further, that Y is a proper subset of $\mathbb{P}^n$ if $n \geq 4$.

vii. Using the previous part or otherwise, show that there is a regular injective map from C into $\mathbb{P}^{n-1}$.

viii. Show that there is a regular injective map of C into $\mathbb{P}^3$.

What's going on? When C is smooth then ψ extends to a regular map $C \times C \longrightarrow \Gamma$ where (x, x) gets mapped to the tangent line at x. The variety X parametrises the secant (& tangent) lines of C. The variety $Y = \pi_1 \pi_2^{-1}(X)$ is called the secant variety and is the union of all secants (& tangents) of C. In this smooth case, the regular injective map you found will actually be an isomorphism of C onto its image.

Secant varieties in $\mathbb{P}^3$ may fill out the whole of $\mathbb{P}^3$ so we cannot extend this argument to embed a projective curve into $\mathbb{P}^2$. In fact, there are projective curves which cannot be embedded (isomorphically) into $\mathbb{P}^2$. Hopefully, I will show you this (or else have you prove it in your exam!).