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MATH5765: Algebraic Geometry, Assignment 1

This assignment is relatively straightforward and you should aim to get full marks for it. I encourage you to discuss the questions with classmates and I certainly hope you check your answers against each other. The definition of cheating for this course means writing something you don't understand.

As always, k is an algebraically closed field.

1. This question is designed to reinforce your understanding of the duality between affine varieties and finitely generated k-algebras which are domains.

Let $R = k[x, y] = k[\mathbb{A}^2]$ and $\sigma : R \longrightarrow R$ be the order two k-algebra automorphism which sends $x \mapsto -x, y \mapsto -y$. Let S be the subring of polynomials f such that $\sigma(f) = f$.

i. Describe explicitly the regular map $s : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ corresponding to the k-algebra homomorphism σ.

ii. Compute S explicitly and show that it is a finitely generated k-algebra which is a domain.

iii. By the previous part, you know $S = k[X]$ for some affine variety. By considering appropriate generators for S or otherwise, show that X is isomorphic to an algebraic subset of $\mathbb{A}^3$ and using this isomorphism to identify X with this subset of $\mathbb{A}^3$ find $\mathcal{I}(X)$.

iv. We see that the inclusion of rings $\iota : S \longrightarrow R$ corresponds to a morphism of varieties $\phi : \mathbb{A}^2 \longrightarrow X$. Using the equality $\sigma \circ \iota = \iota$ or otherwise, show that the fibres of ϕ are stable under s i.e. if F is a fibre then $s(F) = F$.

v. Compute the fibres of ϕ.

2. This question is designed to reinforce your understanding of projective varieties and examine their affine pieces.

Given positive integers $a_0, \dots a_n$ we can form weighted projective space $\mathbb{P}^n(a_0, \dots a_n)$ as the set $k^{n+1} - 0$ modulo the equivalence relation

$$(x_0, \dots, x_n) \sim (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$$

where λ ranges over k^*. Hence $\mathbb{P}^n(1, \dots, 1)$ is just the usual projective space. Write $(x_0; \dots; x_n)$ for the equivalence class containing $(x_0, \dots, x_n)$.

This question concerns the weighted projective plane $X := \mathbb{P}^2(1, 1, 2)$. Let $\iota : X \longrightarrow \mathbb{P}^3$ be the map defined by $(x; y; z) \mapsto (x^2 : xy : y^2 : z)$.

i. Show that the map ι is well-defined and injective.

ii. Show that $Y := \text{im } \iota$ is a closed subset of $\mathbb{P}^3$ and write Y explicitly as the set of zeros of some set of homogeneous polynomials.

iii. There are 4 affine pieces of Y. Show that 2 of them are isomorphic to $\mathbb{A}^2$ and determine the other two (using Q1 if need be).

Remark: In fact, you can generalise the above argument to show that any weighted projective space is a projective variety.