THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

June 2009

MATH3711
Higher Algebra

TIME ALLOWED – 2 hours

TOTAL NUMBER OF QUESTIONS – 4

ANSWER ALL QUESTIONS

THE QUESTIONS ARE OF EQUAL VALUE

ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

THIS PAPER MAY BE RETAINED BY THE CANDIDATE

ONLY CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. [**20 marks**]

a) [**6 marks**] In S_9 (the symmetric group on 9 letters), let

$$x = (123)(456789), \qquad y = (19)(36)(58)(724).$$

i) Find the order of x and y;

ii) Write yxy^{-1} as a product of disjoint cycles and of 2-cycles.

b) [**4 marks**] Let G be a group and H a normal subgroup. Show that

i) for any $x \in G$, $(xH)^n = H$ in the quotient group if and only if $x^n \in H$;

ii) if $[G : H] = m$, then $x^m \in H$ for all $x \in G$.

c) [**5 marks**] Show that a group G is abelian if and only if $(xy)^2 = x^2y^2$ for all $x, y \in G$.

d) [**5 marks**] Let G be a group of order p^2, where p is a prime. Show that G must have a subgroup of order p.

2. [**20 marks**]

a) [**6 marks**] Let G be the group $(\mathbb{C}^*, \cdot)$ of all non-zero complex numbers under usual multiplication, let $\mathbb{R}^+$ be the subgroup of all positive real numbers, and let U be the subgroup consisting of complex numbers $z = x + yi$ with $x^2 + y^2 = 1$ (and $i = \sqrt{-1}$). Use the First Isomorphism Theorem to show that

i) $G/U \cong \mathbb{R}^+$.

ii) $G/\mathbb{R}^+ \cong U$.

b) [**6 marks**] Given that a finite abelian group G is isomorphic to a direct product of primary cyclic groups $\mathbb{Z}/p_i^{m_{ij}}\mathbb{Z}$, where p_i $(1 \le i \le n)$ are prime and m_{ij} $(1 \le i \le n, 1 \le j \le r_{p_i})$ are positive integers, show that G is isomorphic to

$$\mathbb{Z}/h_1\mathbb{Z} \times \mathbb{Z}/h_2\mathbb{Z} \times \cdots \times \mathbb{Z}/h_r\mathbb{Z},$$

where $h_1|h_2|\cdots|h_r$.

c) [**8 marks**] Let D_n ($n \ge 1$ or $n = \infty$) be the dihedral group generated by matrices

$$\sigma = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The dihedral group D_n is called *crystallographic* if $4\cos^2\frac{\pi}{n}$ is an integer.

i) Find all n for which D_n are crystallographic.

ii) For each crystallographic D_n, determine a geometric figure of which D_n is the symmetry group.

3. [**20 marks**] Provide brief explanations to all your answers in the questions below.

a) [**3 marks**] Is $\mathbb{C}/\mathbb{R}$ an algebraic field extension?

b) [**2 marks**] Are $M_2(\mathbb{Z})$ and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ isomorphic rings?

c) [**3 marks**] Is $x^5 - 6x^4 + 6$ irreducible over $\mathbb{Q}$? Is it irreducible over $\mathbb{Q}(i)$?

d) [**3 marks**] Prime factorise 35 in $\mathbb{Z}[i]$. Be sure to check briefly that your factors are indeed prime.

e) [**3 marks**] Is $s := \{p(x) \in \mathbb{C}[x] | \deg p(x) < 5\}$ an ideal of $\mathbb{C}[x]$?

f) [**3 marks**] Let $p(x)$ be the minimal polynomial of $\alpha = \cos \frac{\pi}{48}$ over $\mathbb{Q}$. Is $\deg p(x)$ a power of 2?

g) [**3 marks**] Let $J \lhd \mathbb{R}[x]$. Is every ideal in $\mathbb{R}[x]/J$ principal?

4. [**20 marks**]

a) [**9 marks**]

i) What is the degree $[\mathbb{Q}(\sqrt{3}, \sqrt{5}) : \mathbb{Q}]$?

ii) Show that $\mathbb{Q}(\sqrt{3}, \sqrt{5}) = \mathbb{Q}(\sqrt{3} + \sqrt{5})$.

iii) Show that $\alpha = \sqrt{3} + \sqrt{5}$ is algebraic over $\mathbb{Q}$ and determine its minimal polynomial.

b) [**4 marks**] Simplify $\mathbb{C}[x, y]/\langle x - y^3 \rangle$ by writing it as a quotient of a polynomial ring with fewer variables.

c) [**3 marks**] Consider the function $\nu : \mathbb{C} \longrightarrow \mathbb{R}$ defined by $\nu(z) = |z|^2$. Is ν a euclidean norm on $\mathbb{Z}[i\sqrt{3}]$? Justify your answer.

d) [**4 marks**] Find all commutative rings R which have at least two non-trivial ideals, all of which are maximal. Recall the trivial ideals are 0 and R.