THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

June 2008

MATH3711
HIGHER ALGEBRA

(1) TIME ALLOWED – 3 HOURS

(2) TOTAL NUMBER OF QUESTIONS – 5

(3) ATTEMPT ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. (50 marks total) The following are each worth 5 marks. Justify your answers with a brief explanation (but be careful to mention the key points).

a) Is $10\mathbb{Z}$ a maximal ideal in $\mathbb{Z}$?

b) What is the minimal polynomial of $1+\sqrt{2}$ over $\mathbb{Q}$?

c) Show that x^5+6x^2+3 is irreducible over $\mathbb{Q}$.

d) In this question, we work in the ring $\mathbb{C}[x,y]$. Are the ideals $\langle x+y, y\rangle$ and $\langle 2x, y\rangle$ equal?

e) Let G be the subgroup of $GL_2(\mathbb{R})$ consisting of the 4 matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

G acts on $\mathbb{R}^2$ by the inclusion map $G \hookrightarrow \operatorname{Perm}\mathbb{R}^2$. Find the orbit of $(1,0)$.

f) Is $[\mathbb{Q}(\cos 15^0) : \mathbb{Q}]$ a power of two?

g) Consider a field extension K/F of degree 5. If E is a subfield of K containing F, show that $E = F$ or K.

h) Let R be the ring $\mathbb{R}[x]$ and I be the ideal $\langle x\rangle \cap \langle x^2+1\rangle$. Show that R/I is isomorphic to a product of fields.

i) Suppose G is a group of order 7 acting on a set S via some permutation representation $G \longrightarrow \operatorname{Perm} S$. If S has only 3 elements, show that S has 3 orbits.

j) Describe explicitly the subring $\mathbb{Z}[\frac{1}{2}] \subset \mathbb{Q}$.

2. (12 marks total) For indeterminates x, y, we define the map $\varepsilon : \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x] : p(x,y) \mapsto p(x,0)$.

a) (5 marks) Show that ε is a ring homomorphism.

b) (2 marks) What is the kernel of ε?

c) (2 marks) Show that $\ker\varepsilon$ is a principal ideal.

d) (3 marks) What isomorphism does the first isomorphism theorem give when applied to ε?

3. (10 marks) How many essentially different ways are there of placing a red or green candle in each of 8 equal sectors of a birthday cake? Make sure you show working.

Please see over ...

4. (16 marks) In this question, we let $R = \mathbb{Z}[i]$.

a) (2 marks) Is R a Euclidean domain (no proof necessary) and if so, write down a Euclidean norm for R.

b) (3 marks) Show that $1 + 2i$ is irreducible in $\mathbb{Z}[i]$.

c) (3 marks) Are $1 + 2i$ and $-2 + i$ associates in $\mathbb{Z}[i]$?

d) (5 marks) Prime factorise 15 in $\mathbb{Z}[i]$. Make sure you verify that your factors are indeed irreducible.

e) (3 marks) Prime factorise 15 in $\mathbb{Z}[i\sqrt{2}]$.

5. (12 marks total) Let I, J be ideals of a (not necessarily commutative) ring R.

a) (8 marks) Show that $I + J := \{i + j | i \in I, j \in J\}$ is an ideal of R.

b) (4 marks) Is $\{ij | i \in I, j \in J\}$ always an ideal in R? Either prove it is or provide a counter-example.