School of Mathematics University of New South Wales

MATH3711: Higher Algebra (2007,S1)
Problem Sheet 7 [1]

1. Let R be a UFD and consider a factorisation of $r = p_1 p_2 \dots p_m \in R$ into irreducibles $p_1, \dots, p_m$. Prove the claim in lecture 28 that the divisors of r (i.e. $d \in R$ such that $d|r$) are precisely the products of the p_j's and their associates.

2. Let R be a UFD and $p, q, r \in R$ be such that $pq = r^3$. Prove the claim in lecture 30 that if the greatest common divisor of p, q is 1, then up to associates, p, q are themselves cubes.

3. Let F be any field. Show that the usual factor theorem and remainder theorem hold in $F[x]$. That is, $\alpha \in F$ is a zero of $p(x) \in F[x]$ if and only if $(x-\alpha)$ is a factor of $p(x)$ and more generally, $p(x) = (x-\alpha)q(x)+q(\alpha)$ for some $q[x] \in F[x]$.

4. Which of the following polynomials are prime in $\mathbb{R}[x]$: i) $x^2 + 2$ ii) $x^4 + 4$?

5. Is $x^2 + 2xy + 3y + 3$ prime in $\mathbb{Q}[x]$? What about $\mathbb{Q}(x)[y]$ where $\mathbb{Q}(x)$ is the field of fractions of $\mathbb{Q}[x]$?

6. Is $\mathbb{Z}[x, y]/\langle x^2 + 1\rangle$ a UFD?

7. In $(\mathbb{Z}[i])[x]$ find the greatest common divisor of $5x^2-10x$ and $(1+2i)x$.

8. Let ω be a primitive cube root of unity. Find a minimal polynomial for ω (over $\mathbb{Q}$) and the degree of the extension $\mathbb{Q}(\omega)/\mathbb{Q}$.

9. Let $E/F, K/E$ be field extensions such that K/F is algebraic. Are E/F or K/E algebraic?

10. Is $\alpha = \sqrt{3 + \sqrt{5}}$ algebraic over $\mathbb{Q}$. If so determine its minimal polynomial (use Eisenstein criterion if you like). Can you find subfields of $\mathbb{Q}(\alpha)$ other than $\mathbb{Q}$?

11. Show rigorously that $\mathbb{Q}(\sqrt{3}) \neq \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Hence determine the degree $[\mathbb{Q}(\sqrt{3}, \sqrt{5}) : \mathbb{Q}]$. Using this or otherwise, show that $\alpha = \sqrt{3}+\sqrt{5}$ is algebraic over $\mathbb{Q}$ and determine its minimal polynomial.

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12. Let E/F be a finite field extension of prime degree. Show that E is a simple extension of F.

13. (Not examinable) Show that $\bar{\mathbb{Q}}$ (set of complex numbers algebraic over $\mathbb{Q}$) is countable and so there are uncountably many numbers in $\mathbb{C}$ which are transcendental over $\mathbb{Q}$.

14. Use the Eisenstein criterion to show that $x^3 + 6x + 18$ is irreducible over $\mathbb{Q}[x]$.

15. What's the algebraic closure of $\mathbb{R}$? Is $\mathbb{C}(x) := K(\mathbb{C}[x])$ algebraically closed?

16. Let p be prime. Show that $1 + x + x^2 + \ldots + x^{p-1}$ is irreducible over $\mathbb{Q}$ by summing the geometric progression and changing variables to $y = x - 1$ or otherwise.

17. Can you construct a regular 7-gon (inscribed in unit circle, say by scaling appropriately) using ruler and compass? Hint: The previous question may help.

18. Let $\alpha = \cos \frac{\pi}{48}$. Show that α is algebraic over $\mathbb{Q}$ and that in fact, its minimal polynomial has degree a power of 2.

19. Show that $\mathbb{F}_2[x]/\langle x^2 + x + 1 \rangle$ is the field with 4 elements.