MATH3711: Higher Algebra (2007,S1)
Problem Sheet 6 [1]

1. Show that for any ring R, we have a ring isomorphism $(M_n(R))[x] \simeq M_n(R[x])$ where of course, x is an indeterminate. Remark: I hope the isomorphism is intuitively clear. The best way to write a clinically clean proof is to observe $M_n(R)$ is a subring of $M_n(R[x])$ and then to use an appropriate evaluation ring homomorphism $(M_n(R))[x] \longrightarrow M_n(R[x])$.

2. Is $M_2(M_2(\mathbb{R}))$ isomorphic to a matrix ring over $\mathbb{R}$? (Note: A complete proof of this is probably notationally annoying).

3. For R a ring, show that the diagonal matrices form a subring of $M_n(R)$ which is isomorphic to the product of a number of copies of R.

4. Let $\mathbf{y}_1, \dots, \mathbf{y}_m \in \mathbb{C}^n$ be distinct points and consider the ideal
$$I := \cap_i I(\mathbf{y}_i) \lhd \mathbb{C}[x_1, \dots, x_n].$$
Use the Chinese remainder theorem to show that $\mathbb{C}[x_1, \dots, x_n]/I$ is isomorphic to a product of fields. Hence or otherwise, show that for any $a_1, \dots, a_m \in \mathbb{C}$, you can find a polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ with $p(\mathbf{y}_i) = a_i$ for every i.

5. Describe the field of fractions $K(\mathbb{Z}[i\sqrt{2}])$.

6. Let d be a positive integer with $4|d+1$. Let $\alpha = \frac{1+i\sqrt{d}}{2}$ and consider the ring $R := \mathbb{Z}[\alpha]$. Show that α^2 is a $\mathbb{Z}$-linear combination of α and 1. Show that $S := \mathbb{Z}[i\sqrt{d}]$ is a subring of R and that, as abelian groups, there are two cosets of S in R, namely, S and $\alpha + S$. Hence, sketch on the Argand diagram, the elements of R. Show that the norm function $\nu : \mathbb{C} \longrightarrow \mathbb{R} : z \mapsto |z|^2$ takes integer values on R. Verify lemma lecture 29.

7. (Must do) We look at a special case of the previous question where $\alpha = e^{2\pi i/3} = \frac{1+i\sqrt{3}}{2}$. Show that ν does not take the value 2 on $R :=$

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$\mathbb{Z}[\alpha]$. Show that $r|s$ in R implies $\nu(r)|\nu(s)$ in $\mathbb{Z}$. Hence show that $i\sqrt{3}$ and 2 are irreducible in R. Why do the factorisations

$$2 \times i\sqrt{3} = \frac{3 + i\sqrt{3}}{2} \times (1 + i\sqrt{3})$$

not contradict the fact that $\mathbb{Z}[\alpha]$ is a UFD?

8. Show that $2, 3, 1 \pm \sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$. Hence show $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

9. Using ideas in Q7, show that $1 + 2i$ is prime in $\mathbb{Z}[i]$. Are 5 or 7 prime in $\mathbb{Z}[i]$? If not, find their prime factorisations.

10. Use the Euclidean algorithm of lecture 30 to find the greatest common divisor of: i) $3 + 4i$ and 5 in $\mathbb{Z}[i]$, ii) $6 + 2i, 1 + 3i$ in $\mathbb{Z}[i]$.

11. Compute $\mathbb{Z}[\frac{1}{2}(1 + i\sqrt{11})]^*$