# MATH3711: Higher Algebra (2007,S1)
## Problem Sheet 5 [1]

1. Describe all the subgroups $G$ of $O_2$ of order 16. Your answer should give all subgroups not just subgroups up to isomorphism so in particular, there should be an infinite number of such subgroups. You should probably use the proof of the classification of finite subgroups of $O_2$.

2. Show that the full group $G$ of symmetries of the tetrahedron is $S_4$ as follows. Observe that $G$ embeds naturally in $S_4$. Find a reflection which is a symmetry of the tetrahedron. Hence show that the order of $G$ is 24.

3. Let $G$ be the rotational symmetry group of an icosahedron. How many poles does $G$ have?

4. Let $G$ be a dihedral subgroup of $SO_3$ of order $2n$. How many poles does $G$ have.

5. Describe all possible subgroups of $SO_3$ of order 27.

6. Show that in any ring $R$ that $0r = 0$ for each $r \in R$. Hint: Look in your first year algebra notes. Show that the multiplicative identity in any ring is unique.

7. Let $R$ be a ring. The *centre* of $R$ is defined to be the subset $Z(R) := \{z \in R | zr = rz \text{ for all } r \in R\}$. Show that $Z(R)$ is a subring of $R$. What is the centre of $M_n(\mathbb{R})$?

8. Let $a_0(x), \dots a_n(x) \in \mathbb{C}[x]$. We define the differential operator (with polynomial coefficients)

$$D = \sum_{i=0}^{n} a_n(x)\partial^i \in \text{End}_{\mathbb{C}}\, \mathbb{C}[x]$$

by the formula

$$(Dp)(x) = \sum_{i=0}^{n} a_n(x)\frac{d^i p}{dx^i}(x).$$

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Note that $\partial^0$ is just the identity so $a_0(x)\partial^0$ is just multiplication by $a_0(x)$. Show that in $\mathrm{End}_{\mathbb{C}}\,\mathbb{C}[x]$ we have $\partial x - x\partial = 1$. Show that the set of such differential operators $D$ forms a subring of $\mathrm{End}_{\mathbb{C}}\,\mathbb{C}[x]$.

9. What's $M_n(\mathbb{R})^*$?

10. Let $R$ be a ring and $x \in R$ be such that it has a right inverse $r$ i.e. $rx = 1$ and a left inverse $l$ so that $xl = 1$. Show that $x$ is invertible.

11. Let $I, J \lhd R$. Show that $I + J$ is indeed the ideal generated by $I \cup J$.

12. Let $S$ be a subring of $R$ which is also an ideal. What is $S$?

13. In the ring $\mathbb{C}[x, y, z]$, show that we have equality of the following ideals $\langle x - y, y - z\rangle = \langle x - z, x^2 - xz - y + z\rangle$

14. Let $I$ be the ideal of $\mathbb{Z}[i]$ generated by 5 and $-4+2i$. Find some $z \in \mathbb{Z}[i]$ such that $I$ is generated by $z$. Hint: find a $\mathbb{Z}[i]$-linear combination $z$ of 5 and $-4 + 2i$ with $z$ as "small" as possible. Also, the possibilities for the integer $|z|^2$ are very limited since $5, -4 + 2i \in \langle z\rangle$.

15. Let $Y := \{(0, n) | n \in \mathbb{Z}\} \subset \mathbb{C}^2$. Compute the ideal $I(Y)$.

16. Let $\mathbf{y} \in \mathbb{C}^n$ and $I \lhd \mathbb{C}[x_1, \dots x_n]$ be the ideal generated by $x_1 - y_1, \dots, x_n - y_n$. Show that $x_1^2 x_2 + I = y_1^2 y_2 + I$. Show more generally that for $p \in \mathbb{C}[x_1, \dots, x_n]$ we have $p + I = p(y_1, \dots, y_n) + I$. Hence show that $\mathbb{C}[x_1, \dots x_n]/I \simeq \mathbb{C}$ and that $I = I(\mathbf{y}) = \ker \epsilon_{\mathbf{y}}$.

17. Consider the ideal $\langle y - x^2\rangle \lhd \mathbb{C}[x, y]$. Using the ideas in the previous question or otherwise, show that $\mathbb{C}[x] + \langle y - x^2\rangle = \mathbb{C}[x, y]$. Using the isomorphism theorems as in lecture 23 or otherwise, show that $\mathbb{C}[x, y]/\langle y - x^2\rangle \simeq \mathbb{C}[x]$

18. As in the previous question, simplify the following rings by writing them as a quotient of a polynomial ring with fewer number of variables. i) $\mathbb{C}[x, y, z]/\langle y - z^2, x - y^2\rangle$ ii) $\mathbb{C}[x, y, z]/\langle zy - z, x + y + 4\rangle$ iii) $\mathbb{C}[x, y, z]/\langle y - yx + 4z^3, y + yx + 2x^2\rangle$ iv) $\mathbb{C}[x, y, z]/\langle x + y + z, x + y + 4\rangle$.

19. Compute all the ideals of $\mathbb{C}[x]/\langle x^n\rangle$. Hint: there are $n + 1$ of them.

20. Let $Y \subseteq \mathbb{C}^n$ and $\mathbf{y} \in Y$. Show that $I(\mathbf{y}) \supseteq I(Y)$ and that $I(\mathbf{y})/I(Y)$ is a maximal ideal of $\mathbb{C}[Y]$.

21. Show that $(\mathbb{Z}/n\mathbb{Z})[x] \simeq \mathbb{Z}[x]/\mathbb{Z}[x]n$ as rings (recall $\mathbb{Z}[x]n$ is just the ideal generated by $n$).

22. Find the subring $\mathbb{C}[x^2, x^3]$ of $\mathbb{C}[x]$ generated by $\mathbb{C}, x^2$ and $x^3$. The nicest description is obtained by writing a basis for the underlying complex vector space of $\mathbb{C}[x^2, x^3]$.

23. Show that $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\sqrt{2} + \sqrt{3}]$.