School of Mathematics University of New South Wales

MATH3711: Higher Algebra (2007,S1)
Problem Sheet 4 [1]

1. Use the structure theorem of finitely generated abelian groups in lecture 12 to classify all abelian groups of order 24 up to isomorphism.

2. Let G be a finite product of finite cyclic groups. Using the (weak) Chinese Remainder theorem (exercise 4 of problem sheet 3) instead of the structure theorem for finitely generated abelian groups, show that G is isomorphic to $\mathbb{Z}/h_1\mathbb{Z} \times \ldots \times \mathbb{Z}/h_r\mathbb{Z}$ where $h_1|h_2|\ldots|h_r$.

3. Prove using the theorems of lecture 13, that the group of symmetries of a non-square rectangle in $\mathbb{R}^2$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

4. Let $G = AO_n$, the group of isometries on $\mathbb{R}^n$ and T be the subgroup of translations. Show that $T \lhd G$ and using the third isomorphism theorem, show that G/T is isomorphic to a well known group.

5. Let $S := \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ be the upper half of the complex plane. Let $G = SL_2(\mathbb{R})$. We define a map $\mu : G \times S \longrightarrow S$ as follows. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $z \in S$, let

$$\mu(g, z) = g.z := \frac{az + b}{cz + d}.$$

   Show that $g.z$ is indeed in the codomain S and that μ defines an action of the group G on S.

6. Let S be the set of complex 2×2-matrices and $G = GL_2(\mathbb{C})$. We let G act on S by conjugation as in lecture 16. Let $a = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right), b = \left(\begin{smallmatrix} b_1 & 0 \\ 0 & b_2 \end{smallmatrix}\right) \in S$ be non-scalar matrices. By using the classification of G-orbits given in lecture 15, show that $G.a, G.b$ are isomorphic as G-sets.

7. Generalise the previous question to the following setup. Let S be the set of complex $n \times n$-matrices and $G = GL_n(\mathbb{C})$. We let G act on S by conjugation as before. Let $a, b \in S$ be matrices such that all the eigenvalues for a are distinct and similarly, all the eigenvalues for b are distinct. Show that $G.a, G.b$ are isomorphic G-sets.

[1] by Daniel Chan

8. Let $G = S_n$ act on $\mathbb{R}^n$ by $\sigma.(x_1, \ldots, x_n)^t = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})^t$ as in problem sheet 2 question 24. For $n = 3$ describe all G-orbits and determine which G-orbits are isomorphic and which ones are non-isomorphic.

9. Let $V = \mathbb{R}^n$ and $G = GL_n$. We define an action of G on V by $g.v = g(v)$. Show that $V - \mathbf{0}$ is a G-subset of V isomorphic to G/H for some $H < G$ and compute H explicitly. Is H unique?

10. This question gets you to examine the motivating example of lecture 14. Let G be a group and $\phi : S \longrightarrow T$ a momorphism of G-sets. Show that $\phi(S^G) \subseteq T^G$. Now let $G = \{1, \sigma\}$ and $S, T = \mathbb{R}^2$. Let G act on S, T by $\sigma.(x, y) = (-x, y)$ in S and $\sigma.(x, y) = (-x, -y)$ in T. Hence S, T are the G-sets of the motivating example in lecture 14. Show that S, T are non-isomorphic G-sets by distinguishing their fixed point sets or otherwise.

11. Let $G \leq GL_2(\mathbb{C})$ be the subgroup of diagonal matrices as in lecture 12. Since $GL_2(\mathbb{C})$ is also a subgroup of Perm $\mathbb{C}^2$, the natural inclusion $G \hookrightarrow$ Perm $\mathbb{C}^2$ defines a permutation representation. Describe all the orbits and stabilisers of the corresponding G-set.

12. Let G be a group and X a G-set. Show that for any $J \subset G$, we have $X^J = X^{\langle J \rangle}$. This was the first proposition in lecture 17.

13. Let $G \leq S_5$ be the subgroup generated by $(12), (23), (45)$. The inclusion homomorphism $G \hookrightarrow S_n$ defines a permutation representation. Find all orbits and stabilisers of the corrresponding G-set.

14. How many different ways can you paint the edges of a regular hexagon if each edge may be painted red, white or pink?

15. Show there are 9 099 ways of colouring the faces of a dodecahedron with 3 different colours.