School of Mathematics University of New South Wales

MATH3711: Higher Algebra (2007,S1)
Problem Sheet 2 [1]

1. Find the subgroup of $\mathbb{Z}$ generated by 4 and 6.

2. Find the subgroup of $\mathbb{R}^2$ generated by $(1,0)$ and $(0,1)$.

3. Consider $\sigma \in S_6$ defined using 2 line notation by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}.$$

   Write out σ explicitly as a product of transpositions and hence determine whether it is odd or even. Verify your answer by computing $\sigma\Delta$ where Δ is the difference product.

4. Show that Δ^2 is a symmetric function.

5. Let $f(x_1, \dots, x_n)$ be a complex polynomial. Show that the following two conditions on f are equivalent: i) for any transposition τ we have $\tau.f = -f$ and ii) for any $\sigma \in S_n$ we have $\sigma.f = f$ when σ is even and $\sigma.f = -f$ when σ is odd. Such a polynomial is said to be *anti-symmetric*. Find some examples.

6. There is a right-handed version of all the results in lectures 6. For $H \leq G$, we define a right coset of H in G to be a set of the form $Hg := \{hg | h \in H\}$ for some $g \in G$. Show that G is also a disjoint union of right cosets. (The sophisticated approach is via G^{op}). The set of right cosets is denoted $H \setminus G$. Let $\iota : G \longrightarrow G : g \mapsto g^{-1}$ be the inverse map. It is clearly a bijection. Show that $\iota(Hg) = g^{-1}H$ so ι gives a 1-1 correspondence between $H \setminus G$ and G/H. That is why there is no left or right index.

7. Let G be the symmetric group on 4 symbols S_4 and H be the subset $\{\sigma | \sigma(4) = 4\}$. Show that H is a subgroup. Compute all the left and right cosets of H in G. Verify Lagrange's theorem and the 1-1 correspondence between left and right cosets given in question 5.

8. Let H, K be subgroups of G of order 3 and 5 respectively. Use Lagrange's theorem to show that $H \cap K = 1$.

9. Let G be a group with prime order. Use Lagrange's theorem to find all subgroups of G. Show that G is cyclic.

10. Using the previous exercise or otherwise, find all subgroups of S_3.

11. Show associativity of the subset product claimed in lecture 7 i.e. for subsets K_1, K_2, K_3 of a group G we have $(K_1K_2)K_3 = K_1(K_2K_3)$.

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12. Let $G = \mathbb{C}^*$ and H be the subset of complex numbers of modulus 1. Show that H is a normal subgroup of G and describe the cosets of H. Show that G/H is isomorphic to a subgroup of $\mathbb{R}^*$.

13. Show that $A_n \leq S_n$ is generated by 3-cycles.

14. Let $G = GL_2$ and let H be the subgroup of elements of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ where $a, c \in \mathbb{R}^*$ and $b \in \mathbb{R}$. Compute all the left and right cosets of H in G. If you know some projective geometry you may wish to show that G/H can be naturally identified with the real projective line.

15. Compute explicitly all cosets of $SL_n := \{M \in GL_n | \det M = 1\}$ in GL_n.

16. Compute all cosets of O_2 in GL_2.

17. Let G be a group and H be a subgroup of index two. Show that H is normal.

18. Why is $H = A_n$ normal in $G = S_n$? Find a group isomorphic to G/H.

19. Let $z \in \mathbb{C}^*$ and ϕ be multiplication by z. Is ϕ a group homomorphism from a) $\mathbb{C} \longrightarrow \mathbb{C}$, b) $\mathbb{C}^* \longrightarrow \mathbb{C}^*$?

20. Find all isomorphisms $\phi : \mathbb{Z}/p\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ where p is prime.

21. Isomorphic groups should be identical as far as their group structure is concerned. To illustrate this, consider an isomorphism $\phi : G \longrightarrow G'$. Show

    (a) G is abelian if and only if G' is.

    (b) G, G' have the same order.

    (c) There is a natural bijection between the the subgroups of G and the subgroups of G'. It preserves orders, inclusions and normality.

    (d) If $g \in G$ has order n, so does $\phi(g)$.

22. Show that S_3 and $\mathbb{Z}/6\mathbb{Z}$ both have order 6 (so are isomorphic sets) but are not isomorphic as groups.

23. Fix an integer $n \geq 2$. Suppose that G is a group with distinct elements $\{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}, \tau, \sigma\tau, \ldots, \sigma^{n-1}\tau\}$ where $\sigma^n = 1 = \tau^2$ and $\tau\sigma = \sigma^{-1}\tau$. Show that G is isomorphic to D_n.

24. Find all normal subgroups H of D_n. Show that G/H is dihedral or cyclic. (N.B. This means isomorphic to a dihedral group or cyclic group).

25. For $\sigma \in S_n$ we let $\Phi(\sigma)$ be the linear transformation $\Phi(\sigma) : (x_1, \ldots, x_n)^t \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})^t$. Show that $\Phi : S_n \longrightarrow GL_n$ is a group homomorphism. Determine its image.

26. Show that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the group μ_n introduced in problem sheet 1.

27. Let $f : S \longrightarrow T$ be a bijection of sets. Show that $\phi : \operatorname{Perm} S \longrightarrow \operatorname{Perm} T : \sigma \mapsto f\sigma f^{-1}$ is an isomorphism.

28. Let W be a 2-dimensional subspace of $\mathbb{R}^3$. Recall that $\mathbb{R}^3, \mathbb{R} = \mathbb{R}^1$ are groups with group multiplication given by vector addition and that W is a subgroup of $\mathbb{R}^3$. Show that $\mathbb{R}^3/W$ is isomorphic to $\mathbb{R}$ as a group. (In fact, there is a natural vector space structure on $\mathbb{R}^3/W$ and the isomorphism is even of vector spaces).